SOLUTION OF EQUATIONS OF RELATIVE MOTION ON J2-PERTURBED ECCENTRIC ORBITS

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A new solution is introduced for the relative position and velocity of two spacecraft on eccentric orbits perturbed by Earth oblateness. The equations of relative motion subject to an arbitrary perturbing force are derived in the normalized form suitable for eccentric orbits, with true anomaly as the independent variable. This general framework is used to derive the solution in the case of perturbation by the second zonal harmonic of Earth’s gravitational potential. A comparison of error trends against eccentricity and inter-spacecraft separation is presented between the new solution and several prominent translational and orbital element-based solutions from the literature.

INTRODUCTION

The dynamics of spacecraft relative motion has been the subject of renewed interest as distributed space systems have come to be seen as a mission-enabling technology. Potential commercial and scientific applications for formation-flying spacecraft include on-orbit inspection and servicing, planetary topography and gravity recovery, as well as observations of gravitational waves and direct imaging of extrasolar planets. Due to the limited processing power typical of satellite hardware, computational efficiency can be as important as the accuracy of a dynamics model for onboard implementation. Analytical solutions are particularly valuable because their accuracy is not tied to an integration step size or iteration tolerance, and therefore does not scale directly with computational cost. Solutions may be broadly divided between those based on orbital elements, which capture the underlying physical and geometric properties of the motion, and those using a translational state representation, which closely relates to the system observables. The best-known solution in the latter category is that of Clohessy and Wiltshire (CW), which addresses the linear, time-invariant problem of relative motion between two spacecraft in close proximity on near-circular orbits.1 Shortly after the introduction of CW, London used the method of successive approximations to develop a second-order solution to the circular-orbit problem.2 The most versatile linear solution to the problem of Keplerian relative motion on eccentric orbits was introduced by Yamanaka and Ankersen (YA).3 Recent work by the authors extended YA into a second-order eccentric orbit solution.4 Each of these solutions assumes the only force acting on the spacecraft is due to the central body’s spherically-symmetric gravity field. In reality, space missions are perturbed by interactions with the atmosphere and other celestial bodies, as well as the gravitational effects of Earth’s nonspherical mass distribution. For missions below GEO, the most persistent and significant of these perturbations is the effect of Earth’s oblateness, captured by the \( J_2 \) coefficient in the spherical harmonic representation of the

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gravitational field. Schweighart and Sedwick (SS) used averaging techniques to describe the relative dynamics on near-circular orbits subject to $J_2$ perturbation as a time-invariant system. Butcher recently developed a solution to the time-varying dynamics on $J_2$-perturbed, circular orbits without averaging. The present work builds upon these advances with the introduction of an analytical solution for $J_2$-perturbed relative motion that is valid for arbitrary elliptical orbits.

All relative motion solutions discussed above express the translational state of a deputy spacecraft with respect to a chief spacecraft in the chief’s local-vertical-local-horizontal (LVLH) coordinate frame. This work uses the method of successive approximations to incorporate the effects of $J_2$ perturbation into the second-order eccentric solution previously derived as an extension of YA. First, the perturbing terms in the equations of relative motion for arbitrary chief eccentricity are derived to leading order in $J_2$. These terms arise from four distinct sources. The only real force is due to differential gravitational attraction on the two spacecraft from the oblate mass distribution. Fictitious forces are introduced by the perturbation of the chief’s orbit and therefore the LVLH frame. Because the system dynamics are expressed in normalized coordinates with the chief’s true anomaly as the independent variable, additional terms arise from the effect of $J_2$ on the chief’s true anomaly and orbit eccentricity. The resulting system is reduced to a linear, inhomogeneous system by substituting the first-order YA solution into the perturbation terms. In the case of a circular chief orbit, the inhomogeneous system is time-invariant and readily solved using Laplace Transforms. In the arbitrarily eccentric case, the system is time-varying and a more involved approach must be used. The solution introduced herein was derived through extensive application of the variation of parameters technique on the homogeneous solutions provided by YA. It is convenient to assume that the initial conditions for the perturbation solution are zero, so that the state variables are completely captured by the linear solution at the initial time.

Analysis of the newly derived solution is completed through comparison of its accuracy against other solutions that incorporate the effects of Earth oblateness. For consistency with the underlying assumptions, this comparison uses a numerical integration of the equations of motion subject to $J_2$ as a truth reference. Figure 1 illustrates the performance of the various models for a low earth orbit scenario with a chief eccentricity of 0.1, and modest differences in semimajor axis, eccentricity, and inclination. The relative motion in the orbit plane is shown in the chief’s LVLH frame and the magnitude of the position error is illustrated for four orbit periods. The new solution achieves far better accuracy than the SS model in this scenario due to the latter’s use of averaging in its treatment of $J_2$ and the scenario’s moderate eccentricity. It also offers an improvement in accuracy over a Keplerian propagation, justifying the effort to include the perturbing effects. The last model included in Fig. 1 is based on the mean relative orbit elements (ROE). This model transforms the initial orbital elements from osculating to mean using the mapping of Brouwer and propagates the mean ROE using the analytical solution for the effect of $J_2$. Although this model uses an exact analytical solution and exact transformations between orbital elements and position and velocity vectors, its accuracy is slightly worse than the new solution due to linearization in the mapping between osculating and mean orbital elements.

The paper concludes with a discussion of potential applications for the new solution and future directions for research. A key motivator for higher-order translational-state solutions is the initial relative orbit determination problem using angles-only measurements. Linear models are subject to a range ambiguity which can be resolved through consideration of second-order and perturbation effects. The proposed solution would fit naturally into current research efforts in this area, expanding the applicability of these navigation techniques to include scenarios at any orbit eccentricity.
the new solution itself represents a contribution to the state of the art in modelling spacecraft relative motion, the framework used to incorporate the effects of Earth oblateness could be readily extended to the effects of other perturbations such as atmospheric drag, solar radiation pressure, and third-body gravity.

REFERENCES


