Linear Models for Spacecraft Relative Motion Perturbed by Solar Radiation Pressure

Tommaso Guffanti and Simone D’Amico
Stanford University, Stanford, California 94305

DOI: 10.2514/1.G002822

Many scientific applications require the implementation of spacecraft formation-flying around Earth and other celestial bodies. The control of these formations calls for simple relative dynamics models able to accurately and efficiently incorporate secular and long-periodic effects of relevant perturbations. This paper presents novel linear analytical models of the effect of solar radiation pressure (SRP) on the relative motion of two spacecraft in arbitrary eccentric orbit for two state definitions based on relative orbital elements. The new models are valid both around Earth and around other celestial bodies such as near-Earth asteroids. The linear models are derived by augmenting the state with force model parameters and adjusting the secular and long-periodic effects of SRP. New state transition matrices including $J_2$ and SRP are formalized and validated through comparison with a high-fidelity orbit propagator around Earth and around a near-Earth asteroid. In addition, the dominant trends caused by SRP on the relative orbital elements are captured in closed-form. The new models have accuracy in the range of meter to few tens of meters in all orbit scenarios considered and can be leveraged for formation guidance, control, and design.

I. Introduction

This paper addresses the modeling of secular and long-periodic effects that solar radiation pressure (SRP) has on the spacecraft relative motion in various orbit regimes around different celestial bodies. SRP effect on the relative motion is often neglected by the current literature. This is because SRP becomes significant in orbital scenarios in which formation-flying has not been implemented yet. Nevertheless, future applications will extend formation-flying implementation scenarios, requiring the modeling of this perturbation as well. To date, spacecraft formation-flying has been mainly implemented in low Earth orbit (LEO) by missions such as GRACE [1], TanDEM-X [2], PRISMA [3], and the Canadian Advanced Nanosatellite eXperiment-4 and 5 (CanX-4&5) [4]. Moreover, NASA recently launched the Magnetospheric Multiscale (MMS) mission [5], which includes a formation of four satellites in an elliptical orbit. The variety of scientific applications and the mission costs reduction due to the intrinsic adaptability to spacecraft miniaturization make of great interest the application of multisatellite systems and formation-flying to new orbital scenarios using smaller spacecraft. Relevant examples are given by proposed future missions around Earth as the ESA’s PROBA-3 [6], the Space Rendezvous Laboratory’s (SLAB) Miniaturized Distributed Occultor Telescope (mDOT) [7], on-orbit servicing applications in geostationary orbit (GEO), and others [8]. In addition, a formation or swarm of satellites can be implemented around a near-Earth asteroid (NEA) as the SLAB’s Autonomous Nanosatellites Swarming (ANS) [9] or around planet’s moons [10]. The inclusion of SRP effects is important in orbit scenarios such as Earth’s geostationary orbit or NEA’s orbits, especially for large differential ballistic coefficients between the spacecraft.

Linear dynamic models of the spacecraft relative motion affected by orbital perturbations are well known in literature and have been extensively exploited in real space missions. A comprehensive survey of spacecraft relative motion dynamics models is presented by Sullivan et al. [11]. Using a Cartesian representation of the relative motion, Schweighart and Sedwick [12] and Izzo [13] expand on the Hill–Clohessy–Wiltshire model [14] by including first-order secular effects of $J_2$ and differential drag for formation in near-circular orbits. The Yamanaka–Ankersen model [15] for linear propagation of the relative position and velocity in eccentric orbits does not include perturbations. The inclusion of perturbations is simplified by the use of a relative state with components that are nonlinear combinations of the Keplerian orbital elements of the two spacecraft in formation, which hereafter are called relative orbital elements (ROE). This state varies slowly with time and allows the use of astrodynamics tools such as the Lagrange and Gauss variation of parameters (VOP) form of the equations of motion [16] to include perturbations in closed or semi-analytical form. To date, the literature that exploits this state representation mainly focused on the inclusion of the effects of gravity potential and differential drag, which are the most relevant in the orbital regimes of the currently launched or planned formation-flying missions. In particular, contributions can be divided into two general tracks. The first one originates from the work of Gim and Alfriend that formalized a state transition matrix (STM) including first-order secular and osculating $J_2$ effects on the relative motion in arbitrary eccentric orbit [17]. This STM was used in the design process for NASA’s MMS mission [18] and in the maneuver-planning algorithm of NASA’s CPOD mission [19]. A similar STM was later derived for a fully nonsingular ROE state [20], and more recent works have included higher-order zonal geopotential harmonics [21]. However, this approach has not yet produced an STM including nonconservative perturbations. Meanwhile, researchers at SLAB and collaborators have worked independently to develop models using a different ROE state. Specifically, D’Amico derived an STM that captures the first-order secular effects of $J_2$ on formations in near-circular orbits [22] in his thesis. This model has since been expanded by Gaia et al. [23,24] to include the effect of time-varying differential drag on the relative semi-major axis and on the relative eccentricity vector. This state formulation was first used in flight to plan the GRACE formation’s longitude swap maneuver [25] and has since found application in the guidance, navigation, and control (GN&C) systems of the TanDEM-X [26] and PRISMA [3] missions as well as the AVANTI experiment [27]. More recent work of Koenig et al. [28] presents STM including both first-order secular
SRP models are combined with the inclusion of SRP effects is presented. Fourth, the new linear analytically applying the proposed derivation method. Third, the model-free models of the SRP effect on the considered states are formalized a generic celestial body is presented. Second, the linear analytical presented. For the nonsingular state, a new plant matrix that includes effects of SRP due to both ballistic coefficient differences and orbital position differences.

This paper contributes to the state-of-the-art as described in the following. First, it generalizes the analytical model developed in literature by Cook [32] of the SRP effects on a satellite orbit around Earth to be applied around a generic celestial body. Second, it develops new linear models of the SRP effects in arbitrary eccentric orbit considering two types of relative orbital element states, quasi-nonsingular and nonsingular. In particular, for the quasi-nonsingular state, a new plant matrix that includes effects of SRP due to both ballistic coefficient differences and orbital position differences is presented. For the nonsingular state, a new plant matrix that includes effects of SRP due to just ballistic coefficient differences is presented. In addition, the paper proposes a model-free way to include the SRP effect. It entails the state augmentation by the relative orbital element rates that are more affected by SRP and is a valuable solution when the ballistic coefficient of one of the spacecraft is unknown or very uncertain. The new analytical and model-free SRP plant matrices are combined with $J_2$ models developed by the authors [28] to formalize the new STMs including the joint effect of $J_2$ and SRP. These new STMs are validated and their performance assessed with respect to a high-fidelity numerical propagator both around Earth and around a NEA. Finally, the paper contributes to the state-of-the-art by presenting new reduced closed-form models of the dominant ROE trends caused by SRP in near-circular orbit, both on quasi-nonsingular and on nonsingular states. These closed-form solutions are validated both around Earth, in geostationary orbit, and around a NEA.

After this introduction, the ROE states are defined in Sec. II and the linear models derivation method is presented in Sec. III. A review of the inclusion of Keplerian dynamics and $J_2$ effects on the considered states is presented in Sec. IV, where in particular the original development for the nonsingular $J_2$ model is revisited. Subsequently, Sec. V presents the main contributions of the paper. First, the model of the SRP effect on a single-spacecraft orbit around a generic celestial body is presented. Second, the linear analytical models of the SRP effect on the considered states are formalized applying the proposed derivation method. Third, the model-free inclusion of SRP effects is presented. Fourth, the new linear SRP models are combined with the $J_2$ models to formalize the new $J_2$ + SRP STMs. Fifth, the reduced closed-form solutions of the dominant ROE trends due to SRP are formalized. Finally, in Sec. VI, the developed models are validated by comparison with respect to a high-fidelity orbit propagator both around Earth and around a NEA.

II. Spacecraft Relative Motion State Definition

This paper presents STMs for two ROE states, the quasi-nonsingular and nonsingular. Let $x = [a, e, i, \Omega, \omega, M]^T$ denote the Keplerian orbit elements state. Here, $()^T$ means transpose. For a formation of two spacecraft including a chief (denoted with no subscript) and a deputy (denoted with subscript $d$) the quasi-nonsingular ROE state is defined as

\[ \delta x_{\text{ns}} = \begin{pmatrix} \delta a \\ \delta l \\ \delta e_x \\ \delta e_y \\ \delta i_x \\ \delta i_y \\ (\Omega - \Omega_0) \sin(i) \end{pmatrix} = \begin{pmatrix} (a_d - a) / a \\ (M_d + \omega_d) - (M + \omega) - (\Omega_d - \Omega) \cos(i) \\ e_x d \cos(\omega) - e_x \cos(\omega) \\ e_y d \cos(\omega) - e_y \cos(\omega) \\ i_d - i \\ (\Omega_d - \Omega_0) \sin(i) \end{pmatrix} \]

where $\delta e = (\delta e_x, \delta e_y)^T$ and $\delta i = (\delta i_x, \delta i_y)^T$ are the quasi-nonsingular relative eccentricity and relative inclination vectors, respectively, and $e = (e_x, e_y)^T$ is the quasi-nonsingular eccentricity vector. Here, the mean argument of latitude, $\mu_d = M + \omega$, differs from the true argument of latitude, $\mu = \theta + \omega$, being $M$ and $\theta$ the mean and true anomaly, respectively. The nonsingular ROE state is defined as

\[ \delta x_{\text{n}} = \begin{pmatrix} \delta a \\ \delta l \\ \delta e_x \\ \delta e_y \\ \delta i_x \\ \delta i_y \\ (a_d - a) / a \\ (M_d + \omega_d + f_d \Omega_d) - (M + \omega + f \Omega) \\ e_x d \cos(\omega_d + f_d \Omega_d) - e_x \cos(\omega + f \Omega) \\ e_y d \cos(\omega_d + f_d \Omega_d) - e_y \cos(\omega + f \Omega) \\ \tan(i_d / 2) \cos(\Omega_d) - \tan(i / 2) \cos(\Omega) \\ \tan(i_d / 2) \sin(\Omega_d) - \tan(i / 2) \sin(\Omega) \end{pmatrix} \]

where $f_d$ is an integer parameter and equals either 1 or $-1$. $\delta e^* = (\delta e_x^*, \delta e_y^*)^T$ and $\delta i^* = (\delta i_x^*, \delta i_y^*)^T$ are the nonsingular relative eccentricity and relative inclination vectors, respectively, and $e^* = (e_x^*, e_y^*)^T$ and $i^* = (i_x^*, i_y^*)^T$ are the nonsingular eccentricity and inclination vectors, respectively. The quasi-nonsingular state is so named because it is not uniquely defined when the deputy or chief is in equatorial orbit, either prograde and retrograde. Instead, the nonsingular state is not uniquely defined in retrograde equatorial orbit if $f_d = 1$ or in prograde equatorial orbit if $f_d = -1$. By properly switching the parameter $f_d$ from 1 to $-1$ in close proximity to $i = 180^\circ$ it is possible to retrieve a full nonsingular behavior for all inclinations [16]. Koenig et al. [28] p. 3 present an overview of the relation of the quasi-nonsingular state and of the nonsingular state with $f_d = 1$ with respect to others available in literature. In particular, the quasi-nonsingular state is identical to the D’Amico’s ROE [22], whereas the nonsingular state differs from the differential equinoctial elements employed by Gim and Alfriend [20] by the definition of normalized semi-major axis difference and the use of mean anomaly. The practical use of the nonsingular state with $f_d = -1$ is very rare because equatorial retrograde orbits find no applications around Earth. Nevertheless, for applications around NEAs, retrograde orbits

[1] J. J. 2 effects and differential drag in orbit of arbitrary eccentricity for three different ROE representations, extending and harmonizing the previously mentioned works of Gaia et al. [24] and Gim and Alfriend [17, 20]. Attempts to model the effects of SRP also exist in literature. Spiridonova and Kahle [29] and Spiridonova [30] focused on the on-orbit servicing scenario in GEO and modeled the effects of SRP on the quasi-nonsingular ROE for a formation in near-GEO. The SRP is modeled in the Cartesian Earth-Centered-Inertial (ECI) frame as a differential perturbing acceleration purely dependent on the ballistic coefficient difference between the spacecraft and subsequently applied to the ROE state through linear maps. Following the research track started by D’Amico in his thesis [22], Guffanti et al. [31] presented a linear model for the SRP valid in arbitrary eccentric orbit around Earth including both effects due to spacecraft ballistic coefficient difference and orbital position differences.
are of practical use and setting $f_r = -1$ becomes relevant when approaching $i = 180^\circ$, as presented in Sec. VI.

### III. Derivation Methodology

The methodology used to derive the linear dynamics models for the two different state representations has been developed by the authors in [28,31]. First, the time derivative of the relative state is expressed as

$$\dot{\tau}(t) = f^T \left[ R^T \dot{\tau}(t), \dot{\tau}(t) \right] \dot{\tau}(t), B, \dot{\tau}(t) \right]$$

where $B$ is the ballistic coefficient related to the SRP, $\Delta B = B_j - B$ is the ballistic coefficient difference, and $\dot{\tau}(t)$ is the vector containing the ephemerides of the Sun, as will be better described in Sec. V.

Subsequently, the relative state is augmented by the ballistic coefficient difference, $\dot{\delta\tau}(t), \Delta B^T$, and Eq. (3) is Taylor-expanded to the first order as

$$\begin{align*}
\dot{\delta\tau}(t) &= A \dot{\tau}(t), \dot{\tau}(t) + \mathcal{O}(\delta\tau^2) \\
&= \left[ \frac{\partial A}{\partial \dot{\tau}}_{\dot{\tau}=0} \dot{\tau}(t), \frac{\partial A}{\partial \dot{\tau}}_{\dot{\tau}=0} \dot{\tau}(t) \right] + \mathcal{O}(\delta\tau^2)
\end{align*}$$

(4)

Note that the chain rule derivative provides $\partial A / \partial \dot{\tau} |_{\dot{\tau}=0} = ((\partial A / \partial \dot{\tau}) (\partial \dot{\tau} / \partial \dot{\tau})) |_{\dot{\tau}=0}$ and $\partial A / \partial \dot{\tau} |_{\dot{\tau}=0} = ((\partial A / \partial \dot{\tau}) (\partial \dot{\tau} / \partial \dot{\tau})) |_{\dot{\tau}=0}$. In Eq. (4) the ballistic coefficient difference is assumed constant. This method for deriving the plant is general and has been used in [28,31] to compute plants of various perturbations. The feasibility of solving Eq. (4) in closed form to formalize an STM depends on the nature and structure of the plant matrix. In particular, if the terms of $A$ are constant, the STM is the well-known matrix exponential. If the terms of $A$ are not constant, a linear transformation is sought to transform the plant to a time-invariant form. This transformation has been found for the $J_2$-plant by Koenig et al. [28]. Finding a transformation for SRP is not intuitive; therefore, in Sec. V, a first-order approximation of the SRP STM is developed as

$$\Phi^{\text{SRP}}(\dot{\tau}(t), \dot{\tau}(t), \tau) = I + A^{\text{SRP}}(\dot{\tau}(t), \dot{\tau}(t), \tau) \tau + \mathcal{O}(A^{\text{SRP}})$$

(5)

In particular, the truncation error, which is proportional to $(A^{\text{SRP}}) \tau^2 / 2$, is different from zero when the portion of the plant $\partial A / \partial \dot{\tau} |_{\dot{\tau}=0}$ is included. If only effects proportional to the ballistic coefficient difference are included, the plant retains only the last column and becomes nilpotent. This causes the truncation error to go to zero, and the exact SRP STM is given by $\Phi^{\text{SRP}}(\dot{\tau}(t), \dot{\tau}(t), \tau) = I + A^{\text{SRP}}(\dot{\tau}(t), \dot{\tau}(t), \tau)$ in closed-form, under the assumption of constant orbit elements and Sun ephemerides over the propagation time $\tau$.

### IV. Keplerian Dynamics and Inclusion of $J_2$ Perturbation

The orbital motion around a generic celestial body (CB) with gravity constant $\mu$ and mean radius $R$ is considered. In the absence of perturbations, the variation of the spacecraft orbit elements around the CB is described by Keplerian dynamics as

$$\begin{align*}
\dot{x} \cdot \dot{X} &= \cos(\Omega) \cos(u) - \sin(\Omega) \sin(u) \cos(i) \\
\dot{x} \cdot \dot{Y} &= \sin(\Omega) \cos(u) + \cos(\Omega) \sin(u) \cos(i) \\
\dot{x} \cdot \dot{Z} &= \sin(u) \sin(i) \\
\dot{y} \cdot \dot{X} &= -\cos(\Omega) \sin(u) - \sin(\Omega) \cos(u) \cos(i) \\
\dot{y} \cdot \dot{Y} &= \sin(\Omega) \sin(u) + \cos(\Omega) \cos(u) \cos(i) \\
\dot{y} \cdot \dot{Z} &= \cos(u) \sin(i) \\
\dot{z} \cdot \dot{X} &= \sin(\Omega) \sin(i) \\
\dot{z} \cdot \dot{Y} &= -\cos(\Omega) \sin(i) \\
\dot{z} \cdot \dot{Z} &= \cos(i)
\end{align*}$$

(8)
where \( u = \omega + \theta \) is the true argument of latitude. The position of the spacecraft with respect to the CB is \( r = r \hat{r} \), and can be defined in the CBCI frame as a function of the orbital parameters using Eq. (8) and the spacecraft orbit radius norm given by \( r = a(1 - \varepsilon^2)/(1 + \varepsilon \cos(\theta_s)) \). Similarly, the position of the Sun with respect to the CB can be defined as

\[
\mathbf{r}_s = \mathbf{r}_s(\cos(\Omega_s) \cos(u_s) - \sin(\Omega_s) \sin(u_s) \cos(i_s) \sin(\Omega_s) \cos(u_s) + \cos(\Omega_s) \sin(u_s) \cos(i_s) \sin(\Omega_s) \cos(u_s) - \sin(u_s) \cos(i_s) \sin(\Omega_s) \cos(u_s) + \cos(i_s) \sin(u_s))
\]

where \( u_s = a_0 + \theta_s \) is the true argument of latitude of the Sun and \( r_s = a_s(1 - e_s^2)/(1 + e_s \cos(\theta_s)) \) is the norm of the radius vector connecting the CB to the Sun. Exploiting Eqs. (8) and (9) and the dot product definition, it is possible to express

\[
\begin{align*}
\dot{x} \cdot \mathbf{r}_s &= A \cos(u) + B \sin(u) \\
\dot{y} \cdot \mathbf{r}_s &= -A \sin(u) + B \cos(u) \\
\dot{z} \cdot \mathbf{r}_s &= C
\end{align*}
\]

where the terms \( A, B, \) and \( C \) are as follows [32]:

\[
\begin{align*}
A &= \cos((\Omega - \Omega_s) \cos(u) + \cos(i_s) \sin(u) \sin(\Omega - \Omega_s)) \\
B &= \cos(i_s) \sin(i_s) \sin(u) \sin(\Omega - \Omega_s) + \sin(i_s) \sin(u) \cos(i_s) \sin(u) \sin(\Omega - \Omega_s)) \\
C &= \sin(i_s) \sin(i_s) \cos(u) - \cos(i_s) \sin(u) \cos(u) - \sin(i_s) \sin(u) \cos(u) + \cos(i_s) \sin(u)
\end{align*}
\]

(10)

The force exerted by SRP on the spacecraft is given by [34]

\[
f_{\text{SRP}} = -B \frac{\Psi}{c} \left( \frac{1AU}{r_s} \right)^2 \frac{\mathbf{r}_s}{r_s}
\]

(12)

where the solar flux is \( 1AU \) from the Sun; \( c \) is the light speed, and \( B = Cc/A \) is the spacecraft ballistic coefficient, with \( C \) reflection coefficient, \( A \) cross-sectional illuminated area, and \( m \) spacecraft mass. The force exerted by the SRP on the spacecraft has both in-plane components (radial, \( R_{\text{SRP}} \), and transverse, \( T_{\text{SRP}} \)) and an out-of-plane component \( (N_{\text{SRP}}) \). From Eqs. (10–12), these components are defined as

\[
\begin{align*}
R_{\text{SRP}} &= f_{\text{SRP}} \cdot \mathbf{r}_s - B \frac{\Psi}{c} \left( \frac{1AU}{r_s} \right)^2 (A \cos(u) + B \sin(u)) \\
T_{\text{SRP}} &= f_{\text{SRP}} \cdot \mathbf{r}_s - B \frac{\Psi}{c} \left( \frac{1AU}{r_s} \right)^2 (-A \sin(u) + B \cos(u)) \\
N_{\text{SRP}} &= f_{\text{SRP}} \cdot \mathbf{z} - B \frac{\Psi}{c} \left( \frac{1AU}{r_s} \right)^2 C
\end{align*}
\]

(13)

These three force components are inserted in the Gauss variational equations and averaged over one orbital period assuming that the spacecraft is permanently in sunlight. The resulting orbit element rates due to secular and long-periodic SRP effects around the considered generic CB are

\[
\begin{pmatrix}
\dot{a} \\
\dot{\varepsilon} \\
\dot{\Omega} \\
\dot{i} \\
\dot{\omega} \\
\dot{M}
\end{pmatrix} = \gamma_{\text{SRP}} B \sqrt{a} \begin{pmatrix}
0 \\
-\eta (-A \sin(u) + B \cos(u)) \\
\frac{e \cos(u)}{\eta} C \\
\frac{e \sin(u)}{\eta} C \\
\frac{e \sin(u) \cos(i)}{\eta} C \\
3e + \frac{2}{\eta} (A \cos(u) + B \sin(u))
\end{pmatrix}
\]

(14)

where \( \gamma_{\text{SRP}} = (3/(2\sqrt{\mu}))(\Psi/c)(1AU/r_s)^2 \) is the spacecraft ballistic coefficient, \( \eta = \sqrt{1 - e^2} \), and the terms \( A, B, \) and \( C \) are reported in Eq. (11). More details about the derivation of Eq. (14) are reported in Appendix B. The time derivative of the mean anomaly in Eq. (14) is obtained exploiting the formulation of Hoots reported in [16]. Equation (14) represents the variation of the Keplerian orbit elements due to SRP around a generic CB considering the spacecraft continuously in sunlight. To apply Eq. (14) and the models presented in the following sections to a specific CB, the knowledge of the fictitious orbit parameters of the apparent Sun around the CB is required to be inserted in the terms defined in Eq. (11). For applications around Earth, \( a_0, e_0, \) and \( i_0 \) are, respectively, the Earth’s orbit semi-major axis, eccentricity, and inclination around the Sun. \( \Omega_0 \) equals zero for applications around Earth. \( u_0 \) can be obtained, for example, in simple analytic form using the formulation reported in [35] p. 30. For a CB different from Earth, the Sun ephemerides can be easily obtained by propagation of the CB orbit around the Sun and by converting instantaneous position and velocity of the Sun respect to the CB in orbit elements. It is relevant to note that being \( u_0 \) the true argument of latitude, the presented model is valid for CB in arbitrary eccentric orbit around the Sun, as will be demonstrated in Sec. VI through application of the proposed models around the NEA Eros, which lies on an eccentric orbit around the Sun [36].

B. Linear Relative Motion Dynamics: Analytical Model

The differential SRP effect on the relative motion of two spacecraft in orbit is due to two contributions: the spacecraft ballistic coefficient difference and the difference in orbit position of the two spacecraft as seen from the Sun. The first contribution dominates the second because for real application scenarios the distance between the CB and the Sun is considerably larger than the spacecraft separation, and therefore the difference in orbit position between the spacecraft as seen from the Sun is small. Both these two contributions enter in the SRP plant matrix formulation, which is derived following Sec. III in state-agnostic form as

\[
A_{\text{SRP}} \begin{pmatrix} \chi(t) \chi'_{(0)} \end{pmatrix} = \begin{pmatrix} \frac{\partial \chi}{\partial \xi_{g=0}} \chi(t, \chi'_{(0)}) & \frac{\partial \chi}{\partial \Delta B_{\Delta B=0}} \chi(t, \chi'_{(0)}) \end{pmatrix} \begin{pmatrix} \frac{\partial \chi}{\partial \xi_{g=0}} & 0 \end{pmatrix}
\]

(15)

The portion \( \frac{\partial \chi}{\partial \xi_{g=0}} / \Delta B_{\Delta B=0} \) represents the contribution due to the spacecraft ballistic coefficient difference, whereas the portion \( \frac{\partial \chi}{\partial \xi_{g=0}} / \partial B_{\xi_{g=0}} \) represents the contribution due to the difference in orbit position of the two spacecraft. Being the expressions in Eq. (14) linear in the ballistic coefficient, the expansion with respect to \( \Delta B \) holds for arbitrary large ballistic coefficient differences. On the other hand, the expansion with respect to \( \xi_{g=0} \) requires small ROE with the only exception of \( \delta \lambda \) for the quasi-nonsingular state and \( \delta \Omega \) for the nonsingular state. As stated before, the contribution on the ROE variation due to the term \( \frac{\partial \chi}{\partial \xi_{g=0}} / \Delta B_{\Delta B=0} \) is intuitively larger than the one of \( \frac{\partial \chi}{\partial \xi_{g=0}} / \partial B_{\xi_{g=0}} \). A reduced linear model that accounts
for the pure effect due to the ballistic coefficient difference can be formulated as

\[
\mathbf{A}^{\text{srp,}\Delta B}(\chi(t), \chi'_0(t)) = \begin{bmatrix}
0_{6 \times 6} & \frac{\partial \mathbf{y}^{\text{srp}}}{\partial \Delta B} |_{\Delta B=0}
\end{bmatrix} (\chi(t), \chi'_0(t))
\]  

(16)

In the following sections, both full and reduced plant are formalized for the quasi-nonsingular state, whereas only the reduced plant is formalized for the nonsingular state. The computation of the full quasi-nonsingular plant is facilitated by the fact that Eq. (14) can be easily formulated as a function of the quasi-nonsingular elements and then expanded by \( \partial \chi_{\text{qns}} \). In Sec. VI, the computation of the full quasi-nonsingular plant permits to understand the difference of performance between full SRP model [in Eq. (15)] and reduced SRP model [in Eq. (16)] and to validate the use of the latter.

1. STM Formulation: Quasi-Nonsingular State

\( \mathbf{A}_{\text{qns,full}}^{\text{srp}}(\chi(t), \chi'_0(t)) \) is presented in Appendix C. Using this plant, the complete STM that maps the quasi-nonsingular state from an initial time \( t \) to a final time \( t + \tau \), including Kepler, \( J_2 \), and SRP, is formalized as

\[
\frac{\partial \chi_{\text{qns}}(t + \tau)}{\partial \Delta B} = \Phi_{\text{qns}}^{\text{kep}+\text{srp,full}}(\chi(t), \chi'_0(t), \tau) \left( \frac{\partial \chi_{\text{qns}}(t)}{\partial \Delta B} \right)
\]  

(17)

\[
\Phi_{\text{qns}}^{\text{kep}+\text{srp,full}}(\chi(t), \chi'_0(t), \tau) = \begin{bmatrix}
\Phi_{\text{qns}}^{\text{kep}+\text{srp,full}}(\chi(t), \tau)
\end{bmatrix} + \mathbf{A}_{\text{qns,full}}^{\text{srp}}(\chi(t), \chi'_0(t)) \int_0^{\tau} \Phi_{\text{qns}}^{\text{kep}+\text{srp,full}}(\chi(t), i) \, di
\]  

(18)

In Eq. (18), the series expansion for the SRP STM is truncated at the first order for the reason presented in Sec. III. The truncation error is therefore proportional to \( \mathbf{A}_{\text{qns,full}}^{\text{srp}}(\chi(t), \chi'_0(t), \tau)^2 \). Equation (18) includes the coupling of the effects of \( J_2 \) and SRP and assumes that the chief orbital elements are constant over the propagation interval \( \tau \) as well as the Sun ephemerides. The integral \( \int_0^{\tau} \Phi_{\text{qns}}^{\text{kep}+\text{srp,full}}(\chi(t), i) \, di \) can be computed by integrating the STM in (28) p. 16 or, for small propagation step \( \tau \), approximated accurately by trapezoidal rule as \( \Phi_{\text{qns}}^{\text{kep}+\text{srp,full}}(\chi(t), \tau) + \mathbf{A}_{\text{qns,full}}^{\text{srp}}(\chi(t), \chi'_0(t)) \tau / 2 \). In general, the smaller the \( \tau \), the more accurate is the assumption of constant chief orbital elements and Sun ephemerides over \( \tau \), as well as the use of trapezoidal rule. In a GN&C application scenario, the propagation step \( \tau \) can always be kept small by propagating in parallel the relative motion [using Eq. (18)], the chief absolute motion [using Eqs. (5), (7), and (14)] and the CB motion around the Sun. After every small propagation step \( \tau \), the updated chief elements can be fed into \( A^{\text{srp}} \) and \( \Phi_{\text{qns}}^{\text{kep}+\text{srp}} \), together with the updated Sun ephemerides. In addition, an estimate of the chief state is often available every time new measurements are taken, at that same time that is usually a fraction of the orbit period.

Extracting the last column from \( \mathbf{A}_{\text{qns,full}}^{\text{srp}}(\chi(t), \chi'_0(t)) \), the reduced plant that includes effects proportional to \( \Delta B \) is formalized as

\[
\begin{bmatrix}
\partial \mathbf{e}_x \\
\partial \mathbf{e}_y
\end{bmatrix} = y_{\text{srp}} \sqrt{a} \Delta B \begin{bmatrix}
cos(i) \sin^2 \left( \frac{\Omega - \Omega_e}{2} \right) \sin(u) & -\cos(i) \cos^2 \left( \frac{\Omega - \Omega_e}{2} \right) \sin(u) - \sin(i) \sin(u) \\
\sin^2 \left( \frac{\Omega - \Omega_e}{2} \right) \cos(u) + \cos(i) \cos \left( \frac{\Omega - \Omega_e}{2} \right) \cos(u) - \sin(i) \sin(u)
\end{bmatrix}
\]  

(22)

Using Eq. (19), the STM that maps the quasi-nonsingular state from an initial time \( t \) to a final time \( t + \tau \) including Kepler, \( J_2 \), and SRP is

\[
\begin{bmatrix}
\partial \chi_{\text{qns}}(t + \tau) \\
\partial \Delta B
\end{bmatrix} = \Phi_{\text{qns}}^{\text{kep}+\text{srp,full}}(\chi(t), \chi'_0(t), \tau) \left( \frac{\partial \chi_{\text{qns}}(t)}{\partial \Delta B} \right)
\]  

(19)

Using Eq. (19), the STM that maps the quasi-nonsingular state from an initial time \( t \) to a final time \( t + \tau \) including Kepler, \( J_2 \), and SRP is

\[
\begin{bmatrix}
\partial \chi_{\text{qns}}(t + \tau) \\
\partial \Delta B
\end{bmatrix} = \Phi_{\text{qns}}^{\text{kep}+\text{srp,full}}(\chi(t), \chi'_0(t), \tau) \left( \frac{\partial \chi_{\text{qns}}(t)}{\partial \Delta B} \right)
\]  

(20)

Being \( \Phi_{\text{qns}}^{\text{kep}+\text{srp,full}} \) nilpotent, Eq. (21) is exact assuming that the chief orbital elements are constant over the propagation interval \( \tau \) as well as the Sun ephemerides. There is no coupling between \( J_2 \) and SRP when the reduced model is used. In particular, the integral of the \( \text{kep} + J_2 \) \( \Phi_{\text{qns}}^{\text{kep}+\text{srp,full}}(\chi(t), \chi'_0(t)) \) does not add any effect to the reduced SRP plant, leading to the expression reported at the end of Eq. (21).

Extracting the third and fourth rows from Eq. (19), the STM that maps the quasi-nonsingular state from an initial time \( t \) to a final time \( t + \tau \) including Kepler, \( J_2 \), and SRP is
Equation (23) can be integrated explicitly under the assumption of \( u_0(t) \approx n_0 t + u_0^0 \) (where \( n_0 = \sqrt{\mu_0/\mu^3} \) and \( u_0(t_0) = u_0^0 \)) and constant \( a \) and \( \Omega \) under SRP effect. The resulting closed-form solution is

\[
\begin{align*}
\left( \frac{\delta e_x(t)}{\delta e_y(t)} \right) = & \left( \frac{\delta e_x(t_0)}{\delta e_y(t_0)} \right) + \frac{\gamma_{\text{srp}} \sqrt{\Delta B}}{n_0} \\
& \left( -\cos(i) \sin^2 \left( \frac{t}{2} \right) \left[ \cos(u_0(t) + \Omega - \Omega_0) \right]_t^0 + \cos(i) \cos^2 \left( \frac{t}{2} \right) \left[ \cos(u_0(t) - \Omega + \Omega_0) \right]_t^0 + \sin(i) \sin(i_0) \left[ \cos(u_0(t)) \right]_t^0 \right) \\
& \left( \sin^2 \left( \frac{t}{2} \right) \left[ \sin(u_0(t) + \Omega - \Omega_0) \right]_t^0 + \cos^2 \left( \frac{t}{2} \right) \left[ \sin(u_0(t) - \Omega + \Omega_0) \right]_t^0 \right)
\end{align*}
\]

(23)

which can be specialized for the case of Earth equatorial orbit. Setting \( i \) and \( \Omega_0 \) to zero and considering small \( i \), Eq. (23) becomes

\[
\begin{align*}
\left( \frac{\delta e_x(t)}{\delta e_y(t)} \right) = & \left( \frac{\delta e_x(t_0)}{\delta e_y(t_0)} \right) + \frac{\gamma_{\text{srp}} \sqrt{\Delta B}}{n_0} \cos^2 \left( \frac{i_0}{2} \right) \\
& \left( \cos(n_0 t + u_0^0 - \Omega) - \cos(u_0^0 - \Omega) \right) \\
& \left( \sin(n_0 t + u_0^0 - \Omega) - \sin(u_0^0 - \Omega) \right)
\end{align*}
\]

(24)

Using this plant, the STM that maps the nonsingular state from an initial time \( t \) to an final time \( t + \tau \) including Kepler, \( J_2 \), and SRP is formalized as

\[
\begin{align*}
\left( \frac{\delta \chi(t + \tau)}{\Delta B} \right) = & \Phi^{x+J_2+\text{srp}-\Delta B}_{\text{nonsing}}(\chi(t), \chi_0(t), \tau) \left( \frac{\delta \chi(t)}{\Delta B} \right)
\end{align*}
\]

(26)

For the nonsingular state, the reduced model plant can be formalized inserting Eq. (14) into the definition of \( \delta \chi_0 \) and expanding by \( \Delta B \). The obtained plant is given by

\[
A_{\text{nonsing}}^{x+J_2+\text{srp}-\Delta B}(\chi(t), \chi_0(t)) = \gamma_{\text{srp}} \sqrt{a} \times
\]

\[
\begin{bmatrix}
0 & -e_x \frac{C}{\sin(i)} f_x - \frac{e_y}{\eta} f_y - \frac{e_z}{\eta} f_z \cos(i) \\
\frac{\theta_{\text{x}\times 0}}{0_{\text{x}0}} & 0
\end{bmatrix}
\]

(25)

Being \( A_{\text{nonsing}}^{x+J_2+\text{srp}-\Delta B} \) nilpotent, Eq. (27) is exact assuming that the chief orbital elements are constant over the propagation interval \( \tau \) as well as the Sun ephemerides. There is no coupling between \( J_2 \) and SRP because the integral of the \( x+J_2 \) STM multiplies the block \( \theta_{0x}^{6x1} \) in the nonsingular reduced SRP plant.

This leads to the expression reported at the end of Eq. (27).

As done for the quasi-nonsingular state, extracting the third and fourth rows from Eq. (25) and neglecting terms proportional to \( e^2 \), it is possible to express the nonsingular relative eccentricity vector components rates due to SRP in near-circular orbit as

\[
\begin{align*}
\delta e_x & = \mu_0 = n_0 = \frac{\delta e_x}{t_0} \\
\delta e_y & = n_0 = \frac{\delta e_y}{t_0}
\end{align*}
\]

Fig. 1 Relative eccentricity vector circulation in near-circular Earth equatorial orbit due to SRP (ellipses neglected).
Equation (28) can be integrated explicitly under the assumption of \( u_{\odot}(t) \approx n_{\odot} t + u_{\odot}^0 \) (where \( n_{\odot} = \sqrt{u_{\odot}/a_{\odot}} \) and \( u_{\odot}(t_0) = u_{\odot}^0 \)) and constant \( a \) and \( \Omega \) under SRP effect. The resulting closed-form solution is

\[
\left( \frac{\delta e^*_i(t)}{\delta e^*_i(t_0)} \right) = \frac{\gamma_{\text{sup}} \sqrt{\Delta B}}{n_{\odot}} \left( \begin{array}{c}
-\cos^2 \left( \frac{\Omega}{2} \right) \sin \left[ u_{\odot}(t) + (1 - f_i) \Omega - \Omega_c \right] - \sin^2 \left( \frac{\Omega}{2} \right) \sin \left[ u_{\odot}(t) + (1 + f_i) \Omega - \Omega_c \right] + \\
\cos^2 \left( \frac{\Omega}{2} \right) \cos \left[ u_{\odot}(t) + (1 - f_i) \Omega - \Omega_c \right] + \sin^2 \left( \frac{\Omega}{2} \right) \sin \left[ u_{\odot}(t) + (1 + f_i) \Omega - \Omega_c \right]
\end{array} \right) + \\
\left( \begin{array}{c}
\delta e^*_i(t_0) \end{array} \right)
\]

which can be specialized for the case of Earth equatorial orbit. Setting \( f_i = 1 \) and \( \Omega \) to zero and considering small \( i \), Eq. (29) becomes

\[
\left( \frac{\delta e^*_i(t)}{\delta e^*_i(t_0)} \right) = \left( \frac{\delta e^*_i(t_0)}{\delta e^*_i(t_0)} \right) + \frac{\gamma_{\text{sup}} \sqrt{\Delta B}}{n_{\odot}} \left( \begin{array}{c}
-\cos^2 \left( \frac{\Omega}{2} \right) \sin \left[ u_{\odot}(t) + (1 - f_i) \Omega - \Omega_c \right] - \sin^2 \left( \frac{\Omega}{2} \right) \sin \left[ u_{\odot}(t) + (1 + f_i) \Omega - \Omega_c \right] + \\
\cos^2 \left( \frac{\Omega}{2} \right) \cos \left[ u_{\odot}(t) + (1 - f_i) \Omega - \Omega_c \right] + \sin^2 \left( \frac{\Omega}{2} \right) \sin \left[ u_{\odot}(t) + (1 + f_i) \Omega - \Omega_c \right]
\end{array} \right) \frac{u_{\odot}}{2} \sin \left( \frac{\Omega}{2} \right)
\]

(30)

Similar to the quasi-nonsingular case, Eq. (30) represents a circulation of the nonsingular relative eccentricity vector with radius \( R_{\odot} = (\gamma_{\text{sup}} \sqrt{\Delta B} / n_{\odot}) \cos^2 (i/2) \), rate \( u_{\odot} = n_{\odot} \), initial phase angle \( \theta_{\odot}^0 \), and starting point \( (\delta e^*_i(t_0), \delta e^*_i(t_0)) \), as represented graphically in Fig. 1b.

C. Model-Free Inclusion of Solar Radiation Pressure

The ballistic properties of co-orbiting spacecraft are not always available, especially when approaching unknown resident space objects. Therefore, it is of interest to formalize a model-free SRP STM that does not require knowledge of the spacecraft ballistic coefficients. Because in general the in-plane effects of SRP are larger than out-of-plane, the most simple model-free propagation of the SRP effects can be achieved by augmenting the state by the rates of variation of the relative eccentricity vector components. These rates can be estimated by the navigation filter together with other orbit parameters. Although this model-free propagation neglects effects of SRP on \( \delta t_1 \), \( \delta t_2 \), and \( \delta \theta \), the assumption is critically validated in Sec. VI, where the accuracy results of the model-free model are presented. In general terms, the effect of SRP on \( \delta t_1 \), \( \delta t_2 \), and \( \delta \theta \) could also be included in the model-free propagation by simply augmenting the state by the corresponding rates. Given the presented assumption the augmented quasi-nonsingular state used in model-free propagation is defined as \((\delta \dot{x}_{\text{qns}}^{\text{sup}}, \delta \dot{e}_{\text{qns}}^{\text{sup}}, \delta \dot{e}_{\text{qns}}^{\text{sup}})^T\), whereas the augmented nonsingular state is defined as \((\delta \dot{z}_{\text{qns}}^{\text{sup}}, \delta \dot{e}_{\text{qns}}^{\text{sup}}, \delta \dot{e}_{\text{qns}}^{\text{sup}})^T\). The resulting plant matrix for both quasi-nonsingular and nonsingular states is

\[
\begin{pmatrix}
\Phi_{\text{qns}}^{\text{sup}}(\chi(t), \tau) & \Phi_{\text{qns}}^{\text{sup}}(\chi(t), \tau)
\end{pmatrix}
\]
2. Nonsingular State Model-Free STM

Using Eq. (31), also the STM that maps the nonsingular state from an initial time \( t \) to a final time \( t + \tau \) including Kepler, \( J_2 \), and model-free SRP can be formalized as

\[
\begin{bmatrix}
\delta \chi_{\text{ns}}(t + \tau) \\
\delta e_{x,\text{sup}} \\
\delta e_{y,\text{sup}} \\
\delta \delta a
\end{bmatrix}
= \begin{bmatrix}
\Phi_{\text{ns}}^{\text{kep}+J_2+\text{srp}}(\chi(t), \tau) & 0 \\
0 & I \\
0 & I \\
0 & 0
\end{bmatrix}
\begin{bmatrix}
\delta \chi_{\text{ns}}(t) \\
\delta e_{x,\text{sup}} \\
\delta e_{y,\text{sup}} \\
\delta \delta a
\end{bmatrix}
\] (34)

\[
\Phi_{\text{ns}}^{\text{kep}+J_2+\text{srp}+a}(\chi(t), \tau) = \begin{bmatrix}
\Phi_{\text{ns}}^{\text{kep}+J_2}(\chi(t), \tau) & 0 \times 2^x \\
0 \times 6 & I \times 2^x
\end{bmatrix}
\] (35)

Being \( \Phi_{\text{ns}}^{\text{sup}+a} \) nilpotent and independent from any parameter, Eq. (35) is exact. There is no coupling between \( J_2 \) and SRP, and the structure of the model-free SRP plant leads to the STM formulation reported at the end of Eq. (35).

D. Inclusion of Unmodeled Effects on the Relative Semi-Major Axis

The model presented in Eq. (14) assumes the spacecraft permanently in sunlight. Eclipses occur in a real orbit scenario. In large orbits, eclipse effects are small and introduce small deviations with respect to the analytical solutions presented in the previous sections. An important effect induced by eclipses is a net variation of the orbit semi-major axis. In turn, this variation translates through Kepler in an unmodeled along-track drift of the relative orbit. The error introduced by this along-track drift can be significant and is neglected in the models presented so far. To recover accuracy, it is possible to augment the state with the net rate of relative semi-major axis variation caused by the unmodeled effect. This can be done for both quasi-nonsingular and nonsingular states and for analytic full model, analytic reduced model, and model-free STMs. In state-agnostic form, the resulting analytic full/reduced model STM with state augmented by \( \delta a \) is

\[
\begin{bmatrix}
\delta \chi(t + \tau) \\
\Delta B \\
\delta \delta a
\end{bmatrix}
= \Phi^{\text{kep}+J_2+\text{srp}}(\chi(t), \chi_0, \tau) \\
\begin{bmatrix}
\delta \chi(t) \\
\Delta B \\
\delta \delta a
\end{bmatrix}
\] (36)

\[
\Phi^{\text{kep}+J_2+\text{srp}}(\chi(t), \chi_0, \tau) = \begin{bmatrix}
\Phi^{\text{kep}+J_2}(\chi(t), \chi_0, \tau) & 0 \times 2^x \\
0 \times 6 & I \times 2^x
\end{bmatrix}
\] (37)

In state-agnostic form, the resulting model-free STM with state augmented by \( \delta a \) is

\[
\begin{bmatrix}
\delta \chi(t + \tau) \\
\delta e_{x,\text{sup}} \\
\delta e_{y,\text{sup}} \\
\delta \delta a
\end{bmatrix}
= \Phi^{\text{kep}+J_2+\text{srp}+a}(\chi(t), \tau) \\
\begin{bmatrix}
\delta \chi(t) \\
\delta e_{x,\text{sup}} \\
\delta e_{y,\text{sup}} \\
\delta \delta a
\end{bmatrix}
\] (38)

\[
\Phi^{\text{kep}+J_2+\text{srp}+a}(\chi(t), \tau) = \begin{bmatrix}
\Phi^{\text{kep}+J_2}(\chi(t), \tau) & 0 \times 2^x & 1 \\
0 \times 6 & I \times 2^x & 1
\end{bmatrix}
\] (39)

The augmentation of the state by \( \delta a \) does not introduce any coupling with \( J_2 \); therefore, Eqs. (37) and (39) retain the same level of exactness of the STMs presented in the previous sections.

VI. Validation

In this section the presented STMs are validated with respect to the mean ROE provided by a high-fidelity numerical orbit propagator both around Earth and around NEA Eros. The key parameters and perturbation models for the numerical propagation around the two CB are reported in Tables 1 and 2, respectively. The numerical ground-truth around Earth includes effects of geopotential up to degree and order 10, Sun and Moon gravitational third-body effect, and SRP effect including eclipses [38]. Each simulation around Earth is started on 1 January 2002 at 00:00:00 GPS. The numerical ground-truth around Eros includes effects of gravity potential up to degree and order 10, Sun gravitational third-body effect, and SRP effect including eclipses. The simulation around Eros is started with the NEA mean anomaly around the Sun set at 10° so that the asteroid is closer to the Sun as to maximize the effect of SRP. The physical parameters of the NEA, such as ephemerides, rotation axis orientation with respect to the ecliptic and gravitational parameters are given by [36].

Simulations are initially performed for three nominal test cases presented in Table 3, and then a Monte Carlo analysis that sweeps relevant absolute and relative orbit parameters is presented in the following section. The GEO test represents a formation composed by a massive satellite and a small satellite in Earth geostationary orbit; it is representative of a possible on-orbit servicing scenario in which the small satellite is behind the massive satellite at small radial-normal separation. The GTO test represents the same satellites of the GEO test in high eccentric Earth geostationary transfer orbit, with perigee out of the Earth atmosphere. Finally, the HAO test includes a formation composed by two small spacecraft (a slightly larger mother-ship and deputy) in high-altitude orbit around NEA Eros. The three test cases are chosen such that a model \( J_2 + \text{SRP} \) can reasonably constitute a satisfactory standalone model for orbit propagation. In particular, for the HAO test, the orbit size has been selected large enough such that the effect of higher degree and order gravity potential components (significant for NEAs) does not deteriorate the performance of the \( J_2 + \text{SRP} \) model. At low asteroid

<table>
<thead>
<tr>
<th>Table 1 Numerical orbit propagator parameters around Earth</th>
</tr>
</thead>
<tbody>
<tr>
<td>Integrator</td>
</tr>
<tr>
<td>Step size</td>
</tr>
<tr>
<td>Geopotential</td>
</tr>
<tr>
<td>Solar radiation pressure</td>
</tr>
<tr>
<td>Third-body</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 2 Numerical orbit propagator parameters around near-Earth asteroid Eros</th>
</tr>
</thead>
<tbody>
<tr>
<td>Integrator</td>
</tr>
<tr>
<td>Step size</td>
</tr>
<tr>
<td>Gravity potential</td>
</tr>
<tr>
<td>Solar radiation pressure</td>
</tr>
<tr>
<td>Third-body</td>
</tr>
</tbody>
</table>
orbit, the STMs presented in this paper have to be complemented with models including higher degree and order gravity potential components. Nevertheless, the SRP models formulated in this paper constitute a valid solution to be employed at all NEA orbit altitudes in conjunction with the gravity potential model most appropriate for the specific NEA of interest. Table 4 presents the relevant chief and deputy spacecraft features for the three test cases considered. The resulting ballistic coefficient differences normalized by the chief ballistic coefficient span a range between 50% and 150% and show how large they can be in such realistic formation-flying scenarios. This justifies the emphasis posed in this paper on the modeling of the SRP effect.

Because the presented STMs include only secular effects of $J_2$ and secular/long-periodic effects of SRP, it is necessary to remove short-period oscillations from the time histories of orbit parameters obtained numerically. Because closed-form conversions between osculating and mean states for orbits perturbed by the variety of effects included in the ground-truth are not readily available in the literature, the mean chief and deputy orbit parameters are computed by averaging the osculating ones over a complete orbit. The numerical mean ROE states ($\delta\chi_\text{mean}$) and $\delta\chi_\text{estmean}$ are computed at each time instant from the mean chief and deputy orbit parameters using the state definitions given in Eqs. (1) and (2).

All simulations include an initialization phase of 1 orbit and a propagation phase of 5 orbits. The orbit periods of the three test cases are presented in Table 3. The initialization phase is required to estimate the ROE variation rates used to augment the state for the estimation error is neglected and $\delta\chi_\text{estmean}$ is set equal to the numerical mean ROE state ($\delta\chi_\text{mean}$). The ROE trajectory propagated using the formalized STMs is

$$\delta\chi(t) = \Phi(t)\chi(t),$$

where the state is augmented either with the ballistic coefficient difference or/and the estimated ROE rates according to the specific STMs used. Finally, the error metric used to assess the STMs performance is

$$e_{\delta\chi_j} = \max_{t_i} \frac{\|\delta\chi_j(t) - \delta\chi_j(t_i)\|}{\delta\chi_j(t_i)} \quad t_i \leq t \leq t_f$$

which represents the maximum difference in absolute value between STM and numerical propagation multiplied by the chief mean semi-major axis in order to give dimensionality to the error and provide physical interpretation. Tables 5–7 report the error metric of the GEO test, GTO test, and HAO test, respectively, for both quasi-nonsingular and nonsingular states. Given the chief orbit inclination of the three test cases, the parameter $f_j$ in the nonsingular state is set to 1. Its value is changed while performing the Monte Carlo analysis. The propagation performance of the various STMs $J_2$ + SRP (i.e., analytical full, analytical reduced, and model-free) is assessed and compared against the performance of a $J_2$-only propagation. In GTO, the STM version that accounts for the state augmentation by $\delta\alpha$ is also assessed. Tables 5–7 show clearly the great improvement that modeling SRP introduces on the propagation of the relative eccentricity vector components, both quasi-nonsingular and nonsingular. The error goes from values up to 300 m for a pure $J_2$ propagation to a submeter/meter-level error by including SRP effect either analytically or model-free. Looking at the GEO test in Table 5, no substantial difference is present between analytical

<table>
<thead>
<tr>
<th>Table 3</th>
<th>Initial chief and relative orbits for test cases</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test 1: GEO</td>
<td>42,164</td>
</tr>
<tr>
<td>Test 2: GTO</td>
<td>24,641</td>
</tr>
<tr>
<td>Test 3: HAO</td>
<td>70</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Test 4</th>
<th>Spacecraft features</th>
</tr>
</thead>
<tbody>
<tr>
<td>Test 1: GEO</td>
<td>1000</td>
</tr>
<tr>
<td>Test 2: GTO</td>
<td>1000</td>
</tr>
<tr>
<td>Test 3: HAO</td>
<td>300</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>Table 5</th>
<th>Test 1: GEO (STMs propagation errors for quasi-nonsingular [left], and nonsingular with $f_j = 1$ [right] ROE)</th>
</tr>
</thead>
<tbody>
<tr>
<td>STM</td>
<td>State Aug.</td>
</tr>
<tr>
<td>Kep. + $J_2$</td>
<td>0.11</td>
</tr>
<tr>
<td>Kep. + $J_2$ + SRPfull</td>
<td>0.11</td>
</tr>
<tr>
<td>Kep. + $J_2$ + SRPfull</td>
<td>0.11</td>
</tr>
</tbody>
</table>
Monte Carlo analysis varying the nominal chief orbit inclination of the formation in geostationary orbit: quasi-nonsingular state (left) and nonsingular state $f_r = 1$ (right). Detailed views of the top plots are presented in the bottom plots.

Monte Carlo analysis is presented. The analytical model performs slightly better than the model-free. This is because the model-free STM approximates the variation of the relative eccentricity vector through a constant linear drift instead of a harmonic oscillator as shown in Sec. V. Results from the GTO test case are listed in Table 6 and confirm that the difference between the analytical full and reduced models is negligible. In this case, the model-free STM captures well the effects on the relative inclination vector up to 30 m, which are instead properly modeled by the analytical STMs. In GTO, a substantial improvement of the propagation performance is provided by the augmentation of the state by $\delta \dot{a}$. The unmodeled effects on the relative semi-major axis are up to almost 20 m, causing a drift in along-track separation up to 400 m. This drift is not modeled analytically and introduces a propagation error, which can be successfully avoided through the state augmentation by $\delta \dot{a}$. Results from the HAO test case are listed in Table 7 and show that the analytical full model provides better results than the reduced model in $\delta \dot{a}$ and in the relative inclination vector components. Comparing the analytical and model-free STMs, the effects on the relative eccentricity vector are better captured by the analytical model that accounts for the nonlinear drifting behavior. Further assessment of the STMs is provided in the next section, where a Monte Carlo analysis is presented.

Table 6 Test 2: GTO (STMs propagation errors for quasi-nonsingular [left], and nonsingular with $f_r = 1$ [right] ROE)

<table>
<thead>
<tr>
<th>STM</th>
<th>State Aug.</th>
<th>$\Delta x_{\text{qns}}$</th>
<th>$\Delta x_{\text{ns}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kep. + $J_1$</td>
<td></td>
<td>17.41 354.99 54.80 19.54 29.15 1.48</td>
<td>17.41 355.11 59.42 17.06 14.00 3.63</td>
</tr>
<tr>
<td>Kep. + $J_1$ + SRP$_{\text{full}}$</td>
<td>$\Delta B$</td>
<td>17.41 418.18 2.74 6.84 1.96 1.33</td>
<td>17.41 418.30 2.17 1.73 0.61 0.98</td>
</tr>
<tr>
<td>Kep. + $J_1$ + SRP$_{\text{full}}$</td>
<td>$\Delta B, \delta \dot{a}$</td>
<td>1.86 47.46 2.74 7.35 1.96 1.38</td>
<td>1.86 47.58 2.22 1.47 0.60 1.00</td>
</tr>
<tr>
<td>Kep. + $J_1$ + SRP$_{\text{full}}$</td>
<td>$\Delta B, \delta a$</td>
<td>1.86 47.46 2.73 7.36 1.96 1.38</td>
<td>1.86 47.58 2.22 1.47 0.60 1.00</td>
</tr>
<tr>
<td>Kep. + $J_1$ + SRP$_{\text{nls}}$</td>
<td>$\delta a, \delta \dot{a}$</td>
<td>1.86 23.18 1.32 3.86 29.15 1.80</td>
<td>1.86 23.07 0.95 1.72 13.96 3.80</td>
</tr>
</tbody>
</table>

Table 7 Test 3: HAO (STMs propagation errors for quasi-nonsingular [left], and nonsingular with $f_r = 1$ [right] ROE)

<table>
<thead>
<tr>
<th>STM</th>
<th>State Aug.</th>
<th>$\Delta x_{\text{qns}}$</th>
<th>$\Delta x_{\text{ns}}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Kep. + $J_1$</td>
<td></td>
<td>0.19 13.75 73.20 354.82 1.57 43.90</td>
<td>0.19 66.11 204.73 296.75 37.09 42.31</td>
</tr>
<tr>
<td>Kep. + $J_1$ + SRP$_{\text{full}}$</td>
<td>$\Delta B$</td>
<td>0.19 11.99 9.69 2.40 1.07 18.70</td>
<td>0.19 56.36 2.64 5.13 31.47 25.67</td>
</tr>
<tr>
<td>Kep. + $J_1$ + SRP$_{\text{full}}$</td>
<td>$\Delta B, \delta \dot{a}$</td>
<td>0.19 14.13 36.18 36.03 1.57 43.90</td>
<td>0.19 66.38 55.49 2.69 37.02 42.31</td>
</tr>
<tr>
<td>Kep. + $J_1$ + SRP$_{\text{full}}$</td>
<td>$\delta a, \delta \dot{a}$</td>
<td>0.19 14.13 36.18 36.03 1.57 43.90</td>
<td>0.19 66.38 55.49 2.69 37.02 42.31</td>
</tr>
</tbody>
</table>
A. Monte Carlo Analysis

The metric used to evaluate the performance of the STMs in the Monte Carlo analysis is

$$RMS(\varepsilon_{\theta_j}) = \sqrt{\frac{\sum_{j=1}^{6} \varepsilon_{\theta_j}^2}{6}} \quad (43)$$

which represents the root mean square of the maximum absolute values errors for the ROE state components over the 5 orbits propagation. This metric is chosen because it provides a unique scalar value of performance. The Monte Carlo analysis is developed starting from the presented test cases and varying one specific orbit parameter at a time.

Fig. 3 Monte Carlo analysis varying the nominal chief orbit parameters of the formation in high-altitude orbit around NEA Eros, quasi-nonsingular state.
1. Varying the Nominal Chief Orbit Parameters

Figure 2 reports the RMS error of the $J_2 + \text{SRP}$ STMs for the GEO test varying the inclination of the chief orbit. The performance of the quasi-nonsingular state is illustrated in Fig. 2 (left); the performance of the nonsingular state with $f_r = 1$ is illustrated in Fig. 2 (right). The two plots in Fig. 2 (bottom) present detailed views of the plots in Fig. 2 (top) for inclination close to 0°. Quasi-nonsingular and nonsingular state analytical STMs perform similarly well for inclinations down to 0.5°, with RMS error up to 10 m. At inclinations approaching 0°, as expected, the nonsingular state performs better than the quasi-nonsingular state. In particular, RMS error of 200 m is reached by the quasi-nonsingular state analytical STMs at $i = 0.05°$.

![Graphs showing RMS error vs. inclination and eccentricity](image)

**a)** Results for $f_r = 1$ (left) and for $f_r = -1$ (right). Detailed views of the top plots are presented in the bottom plots.

![Graphs showing RMS error vs. other parameters](image)

**b)** $f_r = 1$

Fig. 4 Monte Carlo analysis varying the nominal chief orbit parameters of the formation in high-altitude orbit around NEA Eros, nonsingular state.
whereas RMS error of 60 m is reached by the nonsingular state analytical STM at $i = 0.001^\circ$. It is relevant to remark that both quasi-nonsingular and nonsingular SRP plant matrices [Appendix C, Eqs. (19) and (25)] includes terms in $1/\sin(i)$ inherited from the model presented in Eq. (14), which itself inherits them from the Gauss variational equations in the classical elements form.

![Fig. 5 Monte Carlo analysis varying the nominal ROE of the formation in geostationary orbit: quasi-nonsingular state (left) and nonsingular state $f_r = 1$ (right).](image1)

![Fig. 6 Monte Carlo analysis varying the nominal ROE of the formation in geostationary transfer orbit: quasi-nonsingular state (left) and nonsingular state $f_r = 1$ (right).](image2)
These terms become singular at \( i = 0^\circ \). Nevertheless, Fig. 2 shows that satisfactory performance is obtained using the nonsingular state down to \( i = 0.01^\circ -0.001^\circ \), which covers the typical range of orbit application scenarios in the Earth geostationary ring. A solution that can be employed at exactly \( i = 0^\circ \) is to use the reduced closed-form solutions proposed in Sec. V for near-circular orbits. Equations (23), (24), (29), and (30) do not suffer from any singularity; however, they model only SRP effect on the relative eccentricity vector components (which is largely the most relevant perturbation in GEO as can be verified from Table 5). The accuracy provided by these closed-form solutions is presented and discussed at the end of this section. The analytical STMs perform better than the model-free STM for both state representations. The reason lies in the fact that the analytical STMs better capture the nonlinear drift of the relative eccentricity vector components and include also effects on other ROE.

Figures 3 and 4 report the RMS error for the HAO test varying the parameters of the chief orbit one at a time and also the NEA Eros mean anomaly around the Sun (\( M_{\text{Ast}} \)). The quasi-nonsingular state is analyzed in Fig. 3. Looking at the three top plots in Fig. 3, it can be noticed how the quasi-nonsingular \( J_2 + \text{SRP} \) analytical STMs perform with RMS error up to 40–50 m for a range of orbit inclinations from \( i = 10^\circ \) to \( i = 170^\circ \). This performance extends down to \( i = 2^\circ \) and up to \( i = 178^\circ \), before it degrades as expected due to the nature of the quasi-nonsingular state. Looking at the four bottom plots in Fig. 3, it can be noticed how the quasi-nonsingular analytical STMs perform well for a broad range of orbit parameters. In particular, the increase of the RMS error increasing the orbit eccentricity is expected because keeping the semi-major axis fixed as in Table 3 and increasing the eccentricity lowers the altitude of the orbit perigee. This causes the increase of effects due to high degree and order gravity potential components, which degrades the

![Fig. 7 Monte Carlo analysis varying the nominal ROE of the formation in high-altitude orbit around NEA Eros: quasi-nonsingular state (left) and nonsingular state \( f_r = 1 \) (right).](image)

![Fig. 8 Trends due to SRP on the relative eccentricity vector components in geostationary orbit for 1 year of propagation: quasi-nonsingular state (left) and nonsingular state (right). Closed-form 1 refers to Eqs. (23) and (29) with \( f_r = 1 \), and closed-form 2 to Eqs. (24) and (30).](image)
J₂ + SRP STMs performance. Varying \( \omega, \Omega, \) and \( M_{\text{At}} \), the RMS error is up to 40 m, showing satisfactory performance of the STMs also when small eclipses occur for specific \( \Omega - M_{\text{At}} \) relative configuration scenarios. The analytical STMs generally provide better results than the model-free one due to reasons mentioned before. The nonsingular state is analyzed in Fig. 4. Looking at Fig. 4a, by switching from \( f_r = 1 \) to \( f_r = -1 \), the nonsingular \( J₂ + \) SRP analytical STM performs well for a broader range of orbit inclinations. By using the analytical STM with \( f_r = 1 \) up to 100° and then switching to \( f_r = -1 \), the RMS error is always kept in the interval 10–50 m, from \( i = 0.001° \) to \( i = 179.999° \). Looking at Fig. 4b, the nonsingular \( J₂ + \) SRP analytical STM (with \( f_r = 1 \)) performs well for a broad range of orbit parameters. Similarly to the quasi-nonsingular state, the increase of the RMS error increasing the orbit eccentricity is expected, because the lower the altitude of the orbit perigee, the greater the effects due to high degree and order gravity potential components. Varying \( \alpha, \Omega, \) and \( M_{\text{At}} \), the RMS error of the analytical STM is up to 40 m, showing satisfactory performance also when small eclipses occur for specific \( \Omega - M_{\text{At}} \) relative configuration scenarios. The analytical STM generally provides better results than the model-free one also for the nonsingular state.

2. Varying the Nominal ROE

After having evaluated the STMs’ performance for various chief orbit parameters, now a similar analysis is done varying the nominal initial ROE for the three test cases analyzed. Figures 5–7 present the RMS error varying the nominal relative semi-major axis and the nominal norm of the relative eccentricity and relative inclination vectors expressed in quasi-nonsingular form. The nominal phase angles of the relative eccentricity and inclination vectors as well as initial \( \delta \) remain the ones set in Table 3. In [22], physical interpretation of the meaning of varying these ROE is provided. In particular, varying the nominal relative semi-major axis entails a variation of the spacecraft radial separation while varying the orbits energy level. Instead, varying the nominal norm of relative eccentricity and relative inclination vectors (expressed in quasi-nonsingular form) entails the variation of the radial/cross-track spacecraft separation without altering the orbits energy level. Looking at Fig. 5 (top), in GEO both quasi-nonsingular and nonsingular STMs perform with RMS error lower than 25 m up to nominal separations of 10 km in \( a_{\delta} \). Looking at Fig. 5 (bottom), quasi-nonsingular and nonsingular state STMs perform with RMS error lower than 40 m up to nominal separations of 20–30 km in \( a[\delta e] \) and \( a[\delta i] \). Looking at Fig. 6, in GTO both quasi-nonsingular and nonsingular STMs perform with RMS error lower than 50 m up to nominal separations of 1 km in \( a_{\delta} \) and up to nominal separations of 5–10 km in \( a[\delta e] \) and \( a[\delta i] \). It is interesting to note that in GTO the model-free STM performs better than the analytic one at larger separations. Looking at Fig. 7, in HAO both quasi-nonsingular and nonsingular STMs perform with RMS error lower than 50 m up to separations of 200 m in \( a_{\delta} \) and up to separations of 1–2 km in \( a[\delta e] \) and \( a[\delta i] \). Therefore, in HAO the proposed linear models hold for tighter formations with respect to Earth. This is expected because the truncation error scales proportional to the ratio spacecraft-separation/orbit-radius, and the orbit radius around a NEA is three orders of magnitude smaller than around Earth.

B. Validation of the Reduced Closed-Form Models

The reduced closed-form models for SRP effect in near-circular orbit elaborated in Sec. V are validated in Figs. 8 and 9 for the GEO and HAO test cases, respectively. In Fig. 8, the closed forms in Eqs. (23) and (29) with \( f_r = 1 \) (labeled closed-form 1) and the closed forms in Eqs. (24) and (30) (labeled closed-form 2) are compared with respect to the numerical ground-truth including only SRP for a propagation of 1 year in GEO. The plots demonstrate how the relative eccentricity vector circulates in 1 year under the effect of SRP as expected. In addition, Eqs. (23) and (29) are in general more accurate than Eqs. (24) and (30) because they introduce less approximating assumptions. Nevertheless, the simplicity of Eqs. (24) and (30) permits to identify an elegant analytical form of the circulation mode.

In Fig. 9, the closed forms in Eqs. (23) and (29) with \( f_r = 1 \) (labeled closed-form 1) are compared with respect to the numerical ground-truth including only SRP for a propagation of 3 months in the HAO test case presented in Table 3 (which has \( i = 100° \)). The plots demonstrate that the trend on the relative eccentricity vector is well captured by the closed-form solutions. In theory, a circulation of the relative eccentricity vector can be predicted also around a NEA if the orbit would remain near-circular. Nevertheless, the effect of SRP causes a relevant variation of the chief orbit eccentricity over one revolution of the NEA around the Sun (almost 2 years long), and at higher eccentricity the reduced closed-form models do not hold anymore. However, over 3 months, the chief orbit eccentricity remains small and the satisfactory results presented in Fig. 9 are obtained.

The presented reduced closed forms provide an immediate and geometrically intuitive evaluation of the most relevant trends introduced by SRP on the ROE. This can be leveraged for quick evaluation of control windows and \( \Delta V \) budgets in a preliminary phase of mission analysis and design of a GN&C system both in GEO and in orbits around NEA or other celestial bodies.

VII. Conclusions

New linear analytical models of the solar radiation pressure (SRP) effect on the spacecraft relative motion in arbitrary eccentric orbit about celestial bodies have been formalized and validated. New \( J₂ + \) SRP state transition matrix (STM) have been derived, which provide accurate relative orbit prediction on a broad range of orbit scenarios. These go from near-circular Earth geostationary orbit to high eccentric Earth geostationary transfer orbit, all the way to orbits.
around near-Earth asteroids (NEAs) in eccentric orbit around the Sun. The generality of the included SRP model makes the presented STMs potentially applicable around a broad class of celestial bodies orbiting the Sun for which the two-body assumption holds. The presented models can be employed for efficient on-board orbit propagation in the framework of a GN&C system, with applications that go from on-orbit servicing in Earth geostationary orbit to NEAs characterization by small satellites swarms. The linearity of the presented models makes them particularly valuable for control purposes, for example, in the framework of optimal impulsive control and model predictive control. Finally, new reduced closed-form solutions of the dominant relative orbit trends caused by SRP are presented. These closed-form solutions can be exploited for quick evaluation of control windows and $\Delta V$ budgets for mission and GN&C system design around various celestial bodies affected by SRP.

Appendix A: $J_2$ STM

A.1. Simplifying Substitutions

$$\eta = \sqrt{1 - e^2}, \quad \kappa = \frac{3fJ_2R^2}{\mu a^2}, \quad G = \frac{1}{\eta}$$  \hspace{1cm} (A1)

$$P = 3 \cos^2(i) - 1, \quad Q = 5 \cos^2(i) - 1, \quad \mathcal{R} = \cos(i), \quad \mathcal{S} = \sin(2i)$$  \hspace{1cm} (A2)

$$\mathcal{U} = \sin(i), \quad \mathcal{W} = \frac{\tan^{-1}(i/2)}{f_{\tau}} \cos^2(i/2)$$  \hspace{1cm} (A3)

$$\omega_t = \omega - \Omega_t, \quad \dot{\omega}_t = \kappa Q, \quad \dot{\Omega}_t = -2 \kappa R, \quad \omega_f = \omega_t + \dot{\omega} \tau$$  \hspace{1cm} (A4)

$$\Omega_f = \Omega_t + \dot{\Omega} \tau$$

$$e_{t,i} = e \cos(\omega_t + f_r \Omega_t), \quad e_{t,f} = e \sin(\omega_t + f_r \Omega_t)$$  \hspace{1cm} (A5)

$$\Omega_{fr} = \Omega_t + \kappa Q \dot{\Omega}_t$$

$$i_{tr} = \tan^{-1}(i/2) \cos(\Omega_t), \quad i_{tf} = \tan^{-1}(i/2) \sin(\Omega_t)$$  \hspace{1cm} (A6)

A.2. Nonsingular State STM

$$\Phi_{kep}^{J_2}(t) = \begin{bmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
\Phi_{21}^{kep+J_2} & 1 & \Phi_{23}^{kep+J_2} & \Phi_{24}^{kep+J_2} & \Phi_{25}^{kep+J_2} & \Phi_{26}^{kep+J_2} \\
\Phi_{21}^{kep+J_2} & 0 & \Phi_{23}^{kep+J_2} & \Phi_{24}^{kep+J_2} & \Phi_{25}^{kep+J_2} & \Phi_{26}^{kep+J_2} \\
\Phi_{21}^{kep+J_2} & 0 & 0 & \Phi_{23}^{kep+J_2} & \Phi_{24}^{kep+J_2} & \Phi_{25}^{kep+J_2} \\
\Phi_{21}^{kep+J_2} & 0 & 0 & 0 & \Phi_{23}^{kep+J_2} & \Phi_{24}^{kep+J_2} \\
\Phi_{21}^{kep+J_2} & 0 & 0 & 0 & 0 & \Phi_{23}^{kep+J_2} \\
\Phi_{21}^{kep+J_2} & 0 & 0 & 0 & 0 & 0 \\
\end{bmatrix}$$  \hspace{1cm} (A7)

$$\Phi_{21}^{kep+J_2} = -\left(\frac{3}{2} n + \frac{7}{2} \kappa (\eta \mathcal{P} + 2 f_r \mathcal{R}) \right) \tau$$  \hspace{1cm} (A8)

$$\Phi_{23}^{kep+J_2} = \kappa e_\tau^* \mathcal{G}(3 \eta \mathcal{P} + 4 Q - 8 f_r \mathcal{R}) \tau$$  \hspace{1cm} (A9)

Appendix B: Solar Radiation Pressure Single-Spacecraft Dynamics

To derive Eq. (14) the authors have elaborated on the formulations presented in the literature by Cook [32] and Hoots (reported in [16]). In particular, the three force components reported in Eq. (13) are inserted in the Gauss variational equations in classical elements form, which are then integrated over the spacecraft true anomaly for that portion of the orbit where the satellite is in sunlight. If the orbit is considered partially in shadow, the integral has to be carried out between the true anomaly of entrance and exit from the sunlight obtaining the equations of net orbit elements variations derived by Cook ([32] pp. 281–283). Here, the force coefficient expressed by Cook as $F$ has been substituted by the authors with $-B(\Psi/c)(1AU/r_e)^2$ ([34] pp. 77–79). If the orbit is considered fully in sunlight, the integral has to be carried out between 0 and $2\pi$, obtaining the averaged equations of orbit elements rates derived by Cook ([32] p. 283) and then complemented by Hoots with the mean anomaly rate ([16] p. 687). These are reported here for completeness:
where $R_{\text{tr}}^p$ and $T_{\text{tr}}^p$ are the radial and transverse force components [Eq. (13)] evaluated at the perigee as if perigee is in sunlight. Afterward, Cook ([32] p. 284) introduces into Eq. (B1) the assumption of Earth orbit and inserts in the force components the Sun ephemerides about the Earth. Instead, the authors are interested in the general expression valid for arbitrary CB and therefore removed this assumption and calculated

$$
\left( \begin{array}{c}
\dot{\alpha} \\
\dot{\epsilon} \\
\dot{i} \\
\dot{\Omega} \\
\dot{\omega} \\
\dot{M}
\end{array} \right) = \left( \begin{array}{c}
0 \\
\frac{3\sqrt{\epsilon^2 - e^2}}{2\alpha} T_{\text{tr}}^p \\
-\frac{3\epsilon \cos(\alpha)}{2\alpha \sqrt{1 - e^2}} N_{\text{tr}} \\
-\frac{3\epsilon \sin(\alpha)}{2\alpha \sqrt{1 - e^2}} N_{\text{tr}} \\
-\frac{3\sqrt{\epsilon^2 - e^2}}{2\alpha} R_{\text{tr}}^p - \dot{\Omega} \cos(i) \\
\frac{9e}{2\alpha} R_{\text{tr}}^p - \sqrt{1 - e^2} (\dot{\omega} + \dot{\Omega} \cos(i))
\end{array} \right)
\tag{B1}
$$

which leads through algebraic manipulations to Eq. (14).

### Appendix C: Solar Radiation Pressure Plant Matrix—Quasi-Nonsingular State Plant

\[
A_{\text{qn}}(\chi(t), \chi_c(t)) = \begin{bmatrix}
A_{1\text{qns}} & A_{2\text{qns}} & A_{3\text{qns}} & A_{4\text{qns}} & A_{5\text{qns}} & A_{6\text{qns}} & A_{7\text{qns}}
\end{bmatrix}
\tag{C1}
\]

#### C.1. First Column $\partial_r/\partial \dot{a}$

The following substitution is employed:

$$
T_1^{a_r} = \frac{\partial \sqrt{a}}{\partial \dot{a}} = 0.5 \sqrt{a}
\tag{C2}
$$

\[
A_{1\text{qns}} = \begin{bmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\tag{C3}
\]

#### C.2. Third Column $\partial_r/\partial \dot{e}_x$

The following substitutions are employed:

$$
T_1^{e_x} = \frac{\partial ((2e^2 + 1 - \eta)/e^2)}{\partial \dot{e}_x} = \frac{(4e_x + e_x/\eta)e^2 - 2e_x(2e^2 + 1 - \eta)}{e^4}
\tag{C4}
$$

#### C.3. Fourth Column $\partial_r/\partial \dot{e}_y$

The following substitutions are employed:

$$
T_1^{e_y} = \frac{\partial ((2e^2 + 1 - \eta)/e^2)}{\partial \dot{e}_y} = \frac{(4e_y + e_y/\eta)e^2 - 2e_y(2e^2 + 1 - \eta)}{e^4}
\tag{C5}
$$
C.4. Fifth Column $\partial_e / \partial \delta i_x$

The following substitutions are employed:

$$
\begin{align*}
T_1^{\delta i} &= \frac{\partial B}{\partial \delta i_x} = -\sin(i)[-\sin(\Omega - \Omega_c) \cos(u_c) \\
&+ \cos(i_c) \sin(u_c) \cos(\Omega - \Omega_c)] + \cos(i) \sin(i_c) \sin(u_c) \\
T_2^{\delta i} &= \frac{\partial C}{\partial \delta i_x} = \cos(i) \sin(\Omega - \Omega_c) \cos(u_c) \\
&- \cos(i_c) \sin(u_c) \cos(\Omega - \Omega_c)] - \sin(i) \sin(i_c) \sin(u_c) \\
T_3^{\delta i} &= \frac{\partial (\cos(i) / \sin(i))}{\partial \delta i_x} = -\frac{1}{\sin(i)^2} \\
T_4^{\delta i} &= \frac{\partial (1 / \sin(i))}{\partial \delta i_x} = -\frac{\cos(i)}{\sin(i)^2}
\end{align*}
$$

(C8)

C.5. Sixth Column $\partial_e / \partial \delta i_y$

The following substitutions are employed:

$$
\begin{align*}
T_1^{\delta i} &= \frac{\partial A}{\partial \delta i_y} = \sin(\Omega - \Omega_c) \cos(u_c) + \cos(i_c) \sin(u_c) \cos(\Omega - \Omega_c) \\
T_2^{\delta i} &= \frac{\partial B}{\partial \delta i_y} = \cos(i) \sin(\Omega - \Omega_c) \cos(u_c) \\
&- \cos(i_c) \sin(u_c) \sin(\Omega - \Omega_c)] \\
T_3^{\delta i} &= \frac{\partial C}{\partial \delta i_y} = \cos(\Omega - \Omega_c) \cos(u_c) + \cos(i_c) \sin(u_c) \sin(\Omega - \Omega_c)
\end{align*}
$$

(C10)

C.6. Seventh Column $\partial_e / \partial \Delta B$

The following substitutions are employed:

$$
\begin{align*}
A_1^{\text{eq}, \text{qns}} &= \begin{pmatrix}
0 \\
A_{21}^{\text{eq}, \text{s}} \\
A_{31}^{\text{eq}, \text{s}} \\
A_{41}^{\text{eq}, \text{s}} \\
A_{51}^{\text{eq}, \text{s}} \\
A_{61}^{\text{eq}, \text{s}}
\end{pmatrix}
= r_{\text{eq}, \text{qns}} B \sqrt{a} \\
A_2^{\text{eq}, \text{qns}} &= \begin{pmatrix}
0 \\
A_{22}^{\text{eq}, \text{s}} \\
A_{32}^{\text{eq}, \text{s}} \\
A_{42}^{\text{eq}, \text{s}} \\
A_{52}^{\text{eq}, \text{s}} \\
A_{62}^{\text{eq}, \text{s}}
\end{pmatrix} \\
A_3^{\text{eq}, \text{qns}} &= \begin{pmatrix}
0 \\
A_{23}^{\text{eq}, \text{s}} \\
A_{33}^{\text{eq}, \text{s}} \\
A_{43}^{\text{eq}, \text{s}} \\
A_{53}^{\text{eq}, \text{s}} \\
A_{63}^{\text{eq}, \text{s}}
\end{pmatrix} \\
A_4^{\text{eq}, \text{qns}} &= \begin{pmatrix}
0 \\
A_{24}^{\text{eq}, \text{s}} \\
A_{34}^{\text{eq}, \text{s}} \\
A_{44}^{\text{eq}, \text{s}} \\
A_{54}^{\text{eq}, \text{s}} \\
A_{64}^{\text{eq}, \text{s}}
\end{pmatrix}
= r_{\text{eq}, \text{qns}} B \sqrt{a} \\
A_5^{\text{eq}, \text{qns}} &= \begin{pmatrix}
0 \\
A_{25}^{\text{eq}, \text{s}} \\
A_{35}^{\text{eq}, \text{s}} \\
A_{45}^{\text{eq}, \text{s}} \\
A_{55}^{\text{eq}, \text{s}} \\
A_{65}^{\text{eq}, \text{s}}
\end{pmatrix} \\
A_6^{\text{eq}, \text{qns}} &= \begin{pmatrix}
0 \\
A_{26}^{\text{eq}, \text{s}} \\
A_{36}^{\text{eq}, \text{s}} \\
A_{46}^{\text{eq}, \text{s}} \\
A_{56}^{\text{eq}, \text{s}} \\
A_{66}^{\text{eq}, \text{s}}
\end{pmatrix}
= r_{\text{eq}, \text{qns}} B \sqrt{a}
\end{align*}
$$

(C11)

\[ \frac{-2^{e_{\alpha} + 1/2}}{c^2} (T_1^{\delta i} + T_2^{\delta i} + T_3^{\delta i} \cos(i)) - \frac{e_{\alpha}}{q} (T_4^{\delta i} + T_5^{\delta i} \cos(i)) \]

(C12)

\[ \frac{-2^{e_{\beta} + 1/2}}{c^2} (T_1^{\delta i} + T_2^{\delta i} + T_3^{\delta i} \cos(i)) - \frac{e_{\beta}}{q} (T_4^{\delta i} + T_5^{\delta i} \cos(i)) \]

(C9)

Acknowledgment

This work was supported by the U.S. Air Force Research Laboratory’s Control, Navigation, and Guidance for Autonomous Spacecraft contract FA9453-16-C-0029. The authors are thankful for their support.

References

