ANGLES-ONLY NAVIGATION FOR AUTONOMOUS ON-ORBIT SPACE SITUATIONAL AWARENESS APPLICATIONS

Joshua Sullivan∗, T. Alan Lovell†, Simone D’Amico‡

This paper addresses the design and validation of a prototype angles-only estimation architecture for on-orbit space situational awareness applications. The proposed navigation approach addresses the main challenges and limitations in the current literature, which generally stem from poor handling of the inherent observability constraints, heavy reliance on prior orbit information, and excessive use of maneuvering to reconcile the unknown range information. The relative orbital element representation is leveraged to decouple the state into weakly and strongly observable components, and to parameterize the relative motion dynamics in slowly time-varying coordinates. Using these features, a novel batch relative orbit estimation algorithm is developed which uses bearing angle trends over at least one orbital period to estimate the state without any prior orbit knowledge. This solution can be further refined using a sequential unscented Kalman filter which exploits nonlinearities in the system dynamics and measurements to disambiguate the unique relative orbit estimate. In particular, a new approach for angles-only filter state propagation based on numerical integration of the Gauss Variational Equations is compared with a filter class using linear state updates. This delineation enables a strategic filter design process to be taken which considers estimation accuracy, efficiency and robustness. Long-term filter stability is compared between maneuver-free and maneuver-inclusive navigation scenarios, and it is shown that sparse control profiles are sufficient to hasten and stabilize the estimation convergence. All algorithms are verified in high-fidelity software-based and hardware-in-the-loop simulation.

INTRODUCTION

The development of future on-orbit space situational awareness (SSA) missions has become an area of growing interest in the space science and engineering community. SSA has long been conducted using coordinated ground-based observation stations to track and maintain a catalog of space objects in Earth orbit. A key example comes from the Space Surveillance Network operated by the United States military, which makes use of more than twenty worldwide optical and/or radar sensing facilities for detecting, imaging, and tracking space objects. Instead, a new generation of space-based SSA assets was initiated with the first Space-Based Surveillance System (SBSS) Pathfinder spacecraft, often abbreviated as SBSS-1. SBSS-1 was declared operational in 2013 by the United States Air Force and is intended to be the first of a constellation of satellites composing a new space-based network for SSA. Unlike their monolithic first-generation counterparts, future on-orbit SSA systems will likely be composed of formations of miniaturized spacecraft platforms which contribute added fault-tolerance and robustness to the SSA network, while also providing improved coverage and resolution by distributing the sensing task over coordinated observers. Accordingly, spacecraft of these envisioned SSA concepts must satisfy increasingly strict requirements on dynamics, guidance, navigation and control accuracy, autonomy in early phases of the mission, and resource efficiency. With these constraints in mind, this paper addresses the development and assessment of a generalized navigation architecture using a vision-based estimation technique known as angles-only navigation to enable future SSA missions in various Earth orbit regimes.

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In angles-only navigation, an observing spacecraft is seeking to estimate the relative orbital state of space objects that appear only as small clusters of pixels in the camera field of view. These pixel centroids intrinsically describe a line-of-sight vector which points from the observer’s camera frame to the target and is subtended by a set of two angles, denoted as the bearing angles. In the context of SSA with space-based observers, the use of vision-based sensing is motivated by the fact that it provides a robust, high dynamic-range, and passive navigation capability which uses simple sensors that are already on board most spacecraft. Furthermore, because of their low cost, low power consumption, and small form factor, vision-based sensors (VBS) enable accurate relative navigation for on-orbit SSA needs while also complementing the current trend of spacecraft miniaturization.

Despite these many advantages, navigation algorithms designed to use angles-only measurements incur several distinct challenges. First and foremost, limited dynamical observability resulting from using 2D bearing angles makes the complete estimation of the 6D relative orbital motion of a target space object difficult or impossible, particularly when using linear dynamics and measurement models. Conducting known orbital maneuvers of the observing spacecraft has become the common solution employed both in the literature and in the low Earth orbit (LEO) flight demonstrations of the ARGON and AVANTI experiments. While maneuvering does generate a known variation in the bearing angle trends that can be exploited to reconcile the unobservable relative range, this approach has the undesirable effect of strongly coupling the achievable relative navigation accuracy with the maneuver-planning task and must be repeated periodically to correct filter estimate divergence. The poor dynamical observability inherent to angles-only navigation also results in a generally heavy reliance on a priori knowledge for initializing any sequential estimation algorithm. While several research studies have revealed that distinct features describing the shape and orientation of the relative trajectory is readily gleaned from angles-only measurements, there has been relatively little research conducted which leverages this unique insight for initial relative orbit determination (IROD). The noteworthy work of Perez et al. shows a promising non-iterative approach for angles-only IROD, but the substantial sensitivity to bearing angle measurement noise and the lack of generalization to inclined and/or eccentric orbits leaves several aspects open for further improvement.

In response to these many challenges, Sullivan et al. developed a maneuver-free approach to angles-only navigation using the unscented Kalman filter (UKF) which was verified for applications in near-circular LEO and highly elliptical orbit (HEO). The method proposed in those works leverages the relative orbital element (ROE) parameterization of the dynamics for several key strengths. First, the weak observability is decoupled into a single state element which well-approximates the range. Second, the filter state propagation is streamlined by using an efficient and accurate linear state transition matrix developed by Koenig et al. which captures secular and long-period effects of the \( J_2 \) perturbation. And third, the dynamical observability is improved by exploiting nonlinearities relating mean to osculating ROE (and then to bearing angles) in the UKF measurement model. These nonlinearities capture separation-dependent features of the relative motion which disambiguate the weakly observable range. The net result is a filtering methodology that accurately converges to an estimate of the full relative motion state without requiring orbital maneuvers. The same key ROE features that were leveraged in the sequential estimation design were also used to explore a new method for efficient and robust filter initialization. In particular, the decoupling of observable and unobservable state elements was used to derive a simple linear method for computing the initial ROE unit vector. While the complete solution for the initial ROE was left subject to a scalar ambiguity, the method shows promise for reducing the computational search space for that unique scaling factor.

This paper builds upon the previous research studies documented by Sullivan et al. by analyzing several aspects of angles-only navigation that were left unaddressed. In particular, this work provides the following contributions to the state-of-the-art: First, a complete semi-analytic approach for batch relative orbit determination (BROD) is developed with flexibility to act as a primary estimator or as a means for initializing a sequential filter. The aforementioned analytic unit vector computation is paired with a simple deterministic method for computing a coarse approximation of the range using the bearing angle geometry over a data batch spanning at least one orbit. The unit vector, in conjunction with this coarse range estimate, is then used to reduce the search space for a regularized batch least-squares numerical solver which refines the coarse ROE estimate over a few iterations. The same BROD algorithm serves the second purpose of running
in parallel with the sequential estimator to aid in filter monitoring and data editing. Second, a method for propagating the ROE using numerical integration of the Gauss Variational Equations (GVE) is derived and implemented in the sequential filter. Numerical propagation increases observability potential since nonlinear dynamical phenomena can be captured and assimilated into estimation updates in the UKF without Taylor series linearization. Furthermore, because the ROE evolve slowly under the effect of perturbations, larger integration time steps can be used to make numerical propagation comparatively more efficient than for rapidly time-varying Cartesian state elements. Whereas the previous filtering approaches exclusively made use of linearized propagation, the method of dynamics modeling can now be more deeply treated as a strategic design choice; potential observability, accuracy, robustness, and efficiency can be traded off for the intended application. Third, the constraint of using a completely "maneuver-free" navigation architecture is relaxed by allowing impulsive orbital reconfiguration to be conducted. The success of the existing maneuver-free sequential filter design obviates complex observability-optimizing maneuver computations in favor of simple single-impulse strategies. Filter convergence accuracy, convergence time, and long-term estimation stability are assessed in the maneuvering case and compared with the maneuver-free counterpart. Finally, the BROD and sequential filtering approaches are verified in high-fidelity simulation. In addition to rigorous software-based numerical validation for both algorithm categories, sequential filtering tests are conducted using an advanced optical testbed which contains a representative spacecraft VBS operating in closed-loop.

Following this introduction, the second section is devoted to modeling preliminaries which focuses on the relative orbital dynamics and the angles-only measurement model. The third section initiates the detailed discussion into the angles-only navigation algorithms by covering major developments in the methods for batch relative orbit determination (BROD). This is followed by the second main algorithm discussion, which pertains to advancements in the sequential relative orbit estimation strategy. Section five contains all high-fidelity validation of the previously highlighted algorithms, with particular focus on software-based and hardware-in-the-loop simulations respectively. Finally, the paper closes with a discussion of lessons learned, conclusions, and ways forward with open research questions.

MODELING PRELIMINARIES

Relative Orbital Dynamics Models

This section provides preliminary fundamentals for modeling the relative motion dynamics between multiple space objects. Rather than providing an exhaustive discussion, the insight provided here is a high-level overview of key concepts that will be tailored to the application of angles-only relative navigation. For a comprehensive resource on spacecraft relative motion dynamics models, the work of Sullivan et al.\cite{Sullivan15} is recommended. In the context of relative orbital motion, the dynamics model parameterizes the trajectory of an orbiting space object with respect to a reference orbit. The reference orbit may describe the physical orbit of another space object or some other meaningful virtual orbit. For this work, the observer spacecraft orbit is treated as the reference about which the target space object motion is described. Irrespective of the dynamical state, the relative orbital mechanics are generally described by a system of nonlinear differential equations

\[ \dot{x}(t) = f(x(t), u(t), t) \] (1)

where \( x(t) \in \mathbb{R}^n \) is the relative motion state, and \( u(t) \in \mathbb{R}^m \) is the control input vector applied by either the observer or the target. The nonlinear equations of motion in Eq. (1) may be linearized about the reference state and control input to produce the linear dynamical state equations

\[ \dot{x}(t) = A(t)x(t) + B(t)u(t) \] (2)

In the above, \( A(t) \in \mathbb{R}^{n \times n} \) denotes the time-variant plant matrix, which captures the unforced dynamics. Instead, the time-variant input sensitivity matrix, \( B(t) \in \mathbb{R}^{n \times m} \), captures the forced dynamics due to \( u(t) \). In this linearized system, \( A(t) \) and \( B(t) \) are simply the Jacobian matrices of the vector-valued function, \( f \), with respect to \( x(t) \) and \( u(t) \), respectively. In general, the nonlinear dynamical model in Eq. (1) must be solved using numerical integration. Instead, it is often possible to find closed form solutions to the model in Eq. (2) which propagate the state from time \( j \) to time \( k \) as

\[ x(t_k) = \Phi_{k,j}x(t_j) + \Upsilon_{k,j}u(t_j) \] (3)
The zero-input response and zero-state response are accounted for via the state transition matrix (STM), \( \Phi \in \mathbb{R}^{n \times n} \), and control convolution matrix (CCM), \( \Upsilon \in \mathbb{R}^{n \times m} \), respectively. The choice of dynamics model (i.e., Eq. 1) and solution methodology (numerical integration or linear propagation via Eq. 3) are major design choices for the estimation architecture and are largely driven by the selection of the dynamical state representation. In the majority of research studies in the literature, the state is parameterized using either a set of translational state components or using relative orbital element.

The relative translational state is made up of the relative position \( \delta\mathbf{r}^R = (x, y, z) \) and relative velocity \( \delta\mathbf{v}^R = (\dot{x}, \dot{y}, \dot{z}) \) of the target defined in the rotating reference frame \( \mathcal{R} \) centered on the observer. The \( \mathcal{R} \) frame is often denoted as the local-vertical/local-horizontal (LVLH) or radial/along-track/cross-track (RTN) frame. It is composed of the orthogonal basis vectors \( \hat{\mathbf{R}} \) directed along the observer absolute position vector, \( \hat{\mathbf{N}} \) in the direction of the observer orbital angular momentum vector, and \( \hat{T} = \hat{\mathbf{N}} \times \hat{\mathbf{R}} \), which completes the right-handed triad. The evolution of this relative state is given by a direct application of Newton’s Law of Gravitation, resulting in the Fundamental Relative Orbital Differential Equations (FRODE).\(^{16}\)

\[
\begin{align*}
\ddot{x} - 2 \dot{f}_o \dot{y} - \ddot{f}_o y - \dot{f}_o x &= \frac{-\mu (r_o + x)}{||\delta\mathbf{r}^R + \mathbf{r}_o^R||^2} + \frac{\mu}{r_o} + \delta d_R \\
\ddot{y} + 2 \dot{f}_o \dot{x} + \ddot{f}_o x - \dot{f}_o y &= \frac{-\mu y}{||\delta\mathbf{r}^R + \mathbf{r}_o^R||^2} + \delta d_T \\
\ddot{z} &= \frac{-\mu z}{||\delta\mathbf{r}^R + \mathbf{r}_o^R||^2} + \delta d_N
\end{align*}
\]

with

\[
||\delta\mathbf{r}^R + \mathbf{r}_o^R||^2 = [(r_o + x)^2 + y^2 + z^2]^\frac{3}{2}
\]

Here, \( r_o \) and \( f_o \) are the radius and true anomaly of the observer, and

\[
\delta d^R \triangleq d_i^R - d_o^R = (\delta d_R, \delta d_T, \delta d_N)^T
\]

is the \( \mathcal{R} \)-frame differential acceleration vector, which accounts for all control inputs and perturbations affecting the spacecraft motion. Hereafter, subscripts \( t \) and \( o \) refer to the target and observer spacecraft, respectively. It is important to note that, even in the Keplerian two-body orbital motion scenario (i.e., \( \delta d^R = 0 \)), Eq. (4) is intractable and no closed-form solutions exist. Instead, by invoking the assumption that the relative separation is small with respect to the observer absolute orbit radius, the differential gravity terms on the right side of Eq. (4) can be linearized. The resulting equations of motion are known as the Lawden or Tschauer-Hempel equations.\(^{17,18}\) While several approaches exist for solving these equations, the STM developed by Yamanaka and Ankersen\(^{19}\) is considered the state-of-the-art model, and is proposed for onboard implementation in the Proba-3 mission.\(^{20}\) If the additional assumption of a circular reference orbit is enacted, the linearized equations reduce further to the well-known Hill-Clohessy-Wiltshire (HCW) equations.\(^{21}\) for which there exists a simple STM solution. Even though this model has a strong flight heritage dating back to the Apollo era, it is largely hampered by a limited range of applicability due to linearization, preclusion of perturbations in the model, and the fast time-variation of the state elements making simple deterministic guidance and control strategies difficult to derive.

As an alternative state, the ROE are made up of combinations of quasi-nonsingular Keplerian orbital elements (OE) of the observer and target spacecraft. For this paper, the vector

\[
\mathbf{oe} = (a, e_x, e_y, i, \Omega, u)^T
\]

constitutes the absolute OE vector of a spacecraft, where \( a \) is the semi-major axis, \( e_x \) and \( e_y \) are the eccentricity vector components, \( i \) is the orbit inclination, \( \Omega \) is the right ascension, and \( u \) is the mean argument of latitude. These OE are only singular for strictly equatorial orbits, where \( \Omega \) is undefined. When talking about OE and ROE, a distinction must be made between osculating and mean elements; the former describe the instantaneous perturbed trajectory, while the latter describe the trajectory subject to an averaged perturbation.
Table 1: Numerical propagation comparative assessment against PRISMA flight products.

<table>
<thead>
<tr>
<th>PRISMA Flight Data</th>
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<tbody>
<tr>
<td>Initial epoch</td>
<td>March 17, 2011 00(^{h}):00(^{m}):00(^{s})</td>
<td></td>
</tr>
<tr>
<td>Duration</td>
<td>6 (hr)</td>
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<table>
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<tr>
<th>Propagator Configuration</th>
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<tbody>
<tr>
<td>Integrator</td>
<td>Runge-Kutta (Dormand-Prince 5(^{th}) order method)</td>
<td></td>
</tr>
<tr>
<td>Geopotential</td>
<td>GRACE Gravity Model GGM01S 120 × 120</td>
<td></td>
</tr>
<tr>
<td>Atmospheric density</td>
<td>NRLMSISE-00 Model</td>
<td></td>
</tr>
<tr>
<td>Solar radiation pressure</td>
<td>Flat plate with canonical Earth shadow</td>
<td></td>
</tr>
<tr>
<td>Third-body gravity</td>
<td>Lunar and solar point masses; analytic ephemerides</td>
<td></td>
</tr>
</tbody>
</table>

effect. To avoid confusion, the notation \(\tilde{\cdot}\) will be used to denote an osculating quantity. The evolution of the OE state comes from the Gauss Variational Equations (GVE), where the time variation of the osculating OE due to an acceleration vector described in the local frame of the spacecraft is given by

\[
\frac{d\tilde{\alpha}_i}{dt} = G(\tilde{\alpha}_i) d^R
\]

with \(G \in \mathbb{R}^{6\times3}\) denoting the well-documented GVE matrix.\(^{16}\) This set of absolute OE variational equations can be used now to derive a set of variational equations for the osculating ROE, \(\delta \tilde{\alpha}_i\). First, let the ROE be defined functionally as some linear or nonlinear combination of absolute orbital elements, \(\delta \tilde{\alpha}_i = g(\tilde{\alpha}_i, \tilde{\alpha}_t)\).

Then, the time evolution of the ROE simply comes from differentiating the vector-valued function \(g\) with respect to time and substituting the individual contributions from the GVE for each spacecraft, yielding

\[
\frac{d\delta \tilde{\alpha}_i}{dt} = \left[ \frac{\partial g}{\partial \tilde{\alpha}_o} G(\tilde{\alpha}_o) + \frac{\partial g}{\partial \tilde{\alpha}_t} G(\tilde{\alpha}_t) \right] d^R_o + \left[ \frac{\partial g}{\partial \tilde{\alpha}_t} G(\tilde{\alpha}_t) \right] \delta d^R_i
\]

The rotation matrix \(R^t_i\) transforms \(d^R_i\) from \(R\) to the targets RTN frame, \(R^t_i\), in order to apply the GVE. Notice from Eq. (8) that the total ROE evolution is intuitively comprised of variations due to the perturbation of the reference orbit and variations due to differential accelerations in the observer-target system. While Eq. (4) and Eq. (8) fundamentally describe the same underlying physics, the effectiveness of the ROE state is evident when considering again the Keplerian two-body example. Unlike with the FRODE, the solution of Eq. (8) in this case is trivial; for orbits of equal energy the ROE are all constant. Furthermore, under the effect of perturbations, the osculating ROE evolve slowly in time with respect to the reference orbital period and can be decomposed into short-period (on the order of the orbital period) oscillations, long-period (an order of magnitude larger than the orbital period) oscillations, and secular variations. This slow time-variation allows larger time steps to be used when numerically integrating the GVE, effectively increasing propagation efficiency for comparable modeling accuracy. Instead, averaging theory can be effectively applied to obtain the variational equations for the mean ROE subject to the secular perturbation effects. This method leads to tractable equations of motion which can be solved in closed-form, as demonstrated by both Koenig et al.\(^{14}\) for \(J_2\) and differential atmospheric drag perturbations, and Guffanti et al.\(^{22}\) for solar radiation pressure (SRP) and lunisolar third-body perturbations. Within an angles-only navigation filter, these two models enable accurate and efficient modeling of the perturbation environment in LEO and GEO respectively, and their combination can be used to bridge the gap between low and high altitude phases of HEO scenarios.

As evidence for the GVE propagation efficiency, a numerical integration assessment has been conducted to study the performance of the two classes of dynamics models as a function of integration step size. Precise orbit determination products from the PRISMA mission\(^{23}\) act as the "ground truth" data by which the outputs of the numerical propagators are evaluated. For the orbital element case, the absolute GVE in Eq. (7) are numerically integrated; for the translational element case, the absolute Fundamental Orbital Differential Equations (FODE) are numerically integrated. In both cases, high-fidelity models of Earth geopotential, atmospheric drag, SRP, and lunisolar third-body perturbations are used (see Table 1). Using 200 evenly-spaced
initial condition sets from within the flight data, the orbits are propagated with integration step size varied in the set $\Delta t \in \{10, 30, 60, 120\}$ sec. The mean 3D root-mean-squared (RMS) position propagation errors and 1-$\sigma$ standard deviations are computed from the 200 sample trajectories at each timestep and shown in Figure 1. It is important to note that similar trends to those depicted are also found for the velocity propagation errors. Clearly, numerically integrating the GVE allows for superior propagation accuracy while using a step size that is an order of magnitude greater than the FODE-based test. In the angles-only navigation scenario, where measurements come at the sparse rate of 1 - 10 minutes, a larger integration step size means less overall calls to the state update algorithm and a corresponding increase in filter propagation efficiency.

In general, the choice of ROE function $g$ is dependent on the orbital scenario. For this paper, the ROE are defined as the following combination of quasi-nonsingular absolute orbital elements:

$$\delta \mathbf{e} = \begin{pmatrix} \delta a \\ \delta \lambda \\ \delta e_x \\ \delta e_y \\ \delta i_x \\ \delta i_y \end{pmatrix} = \begin{pmatrix} \delta a \\ \delta \lambda \\ |\delta e| \cos \varphi \\ |\delta e| \sin \varphi \\ |\delta i| \cos \vartheta \\ |\delta i| \sin \vartheta \end{pmatrix} \triangleq \begin{pmatrix} (a_t - a_o)/a_o \\ u_t - u_o + (\Omega_t - \Omega_o) \cos i_o \\ e_{x,t} - e_{x,o} \\ e_{y,t} - e_{y,o} \\ i_t - i_o \\ (\Omega_t - \Omega_o) \sin i_o \end{pmatrix}$$

Specifically, $\delta a$ is the relative semi-major axis, $\delta \lambda$ is the relative mean longitude, $\delta e$ is the relative eccentricity vector, and $\delta i$ is the relative inclination vector. In the second form of the ROE in Eq. (9), the relative eccentricity and inclination vectors have been written in polar form in terms of their magnitudes and phase angles, $\varphi$ and $\vartheta$, respectively. The former angle represents the relative periapsis, while the latter represents the relative ascending node. Just as with the absolute orbital elements defined in Eq. (6), this definition of the ROE is only singular for strictly equatorial orbits, and is equally applicable to osculating and mean elements.

This definition of mean quasi-nonsingular ROE in particular has a powerful geometric interpretation in near-circular orbits wherein each element corresponds to distinct features of the relative trajectory in the RTN frame. Specifically, $\delta a$ and $\delta \lambda$ capture mean offsets in the radial and along-track directions respectively, the magnitudes of $\delta e$ and $\delta i$ correspond respectively to the magnitudes of the in-plane (radial/along-track) oscillations and out-of-plane (cross-track) oscillations, and their phase angles $\varphi$ and $\vartheta$ dictate the orientation and warping of the tilted ellipse in the radial/cross-track plane (see Figure 2a). In the context of far-range angles-only navigation, $\delta \lambda$ is a strong approximation to the weakly observable range. D’Amico 24,25 leveraged these features to formulate the so-called relative eccentricity/inclination-vector separation guidance laws.
which are useful for establishing passively safe relative trajectories in the presence of along-track position uncertainty. The near-circular orbit geometric interpretation was extended to arbitrarily eccentric orbits by Sullivan et al.\textsuperscript{13} by the introduction of a modified combination of the mean quasi-nonsingular ROE denoted by \((\delta a, \delta \lambda^*, \delta e^*, \delta i)^T\). In addition to aforementioned geometric features, the new interpretation superimposes oscillations in the radial and along-track directions that are proportional to the observer orbit eccentricity, \(e_o\). Figure 2b shows the dimensionless RTN frame geometry, with components proportional to \(e_o\) shown in red.

### Angles-Only Measurement Model

The angles-only measurement model describes the relationship between the internal dynamical state used in the filter and the measurements received by the VBS. As previously mentioned, this work uses an internal state composed of the quasi-nonsingular ROE defined in Eq. (9). The VBS measurements at each sample time are a pair of synchronous bearing angles, denoted as the azimuth \(\alpha\) and the elevation \(\epsilon\). These bearing angles can be interpreted in several ways. From a purely geometric standpoint, the bearing angles subtend a line-of-sight (LOS) vector in the observers VBS frame, \(V\), which points from the observer to the target. Instead, one might also consider these angles as an analog for the image frame coordinates to the target object pixel cluster. The former interpretation is highlighted in Figure 3, where the \(\hat{V}\) and \(R\) frames are shown, and the bearing angles are defined with respect to the VBS-frame relative position vector, \(\delta r^V\). Formally, the bearing angle measurement model is written with respect to \(\delta r^V\) as

\[
y = h^\prime (\delta r^V) = \begin{pmatrix} \alpha \\ \epsilon \end{pmatrix} = \begin{pmatrix} \arcsin (\delta r^V_\gamma / ||\delta r^V||) \\ \arctan (\delta r^V_x / \delta r^V_z) \end{pmatrix}
\]

(10)

Figure 3: Bearing angle geometry in the vision-based sensor frame, \(\hat{V}\).
In circular orbits (where the basis vector pointing in the along-track direction, \( R \), leveraged to disambiguate the weakly observable inter-spacecraft range; the measurement model obviously contains several distinct sources of useful nonlinearities which can be nonlinear and retains the curvature of the trajectories inherently captured by the orbital element description. Forming which is nonlinear in terms of separation and linear with respect to \( J \), important to convert to osculating elements if possible (see Algorithm 1, Line 4). This ensures that the bearing angles (which live in an inherently “osculating” space) are matched to properly oscillating ROE trends. Schaub and Junkins\(^{26}\) and Alfriend et al.\(^{16}\) both provide an analytical model for the mean-to-osculating transformation which is nonlinear in terms of separation and linear with respect to \( J_2 \). Instead, there are limited closed-form models for other perturbation sources currently available in the literature. Second, in order to compute the relative position vector (or LOS) vector, it is first necessary to transform the absolute orbital elements to absolute inertial position and velocity components, as in Algorithm 1, Line 6. This transformation is transforming from mean or osculating ROE to bearing angles is highlighted in Algorithm 1. There are two particularly important components to this model that deserve attention. First, when using mean elements it is important to convert to osculating elements if possible (see Algorithm 1, Line 4). This ensures that the bearing angles (which live in an inherently “osculating” space) are matched to properly oscillating ROE trends. Schaub and Junkins\(^{26}\) and Alfriend et al.\(^{16}\) both provide an analytical model for the mean-to-osculating transformation which is nonlinear in terms of separation and linear with respect to \( J_2 \). Instead, there are limited closed-form models for other perturbation sources currently available in the literature. Second, in order to compute the relative position vector (or LOS) vector, it is first necessary to transform the absolute orbital elements to absolute inertial position and velocity components, as in Algorithm 1, Line 6. This transformation is nonlinear and retains the curvature of the trajectories inherently captured by the orbital element description. The measurement model obviously contains several distinct sources of useful nonlinearities which can be leveraged to disambiguate the weakly observable inter-spacecraft range;\(^{13}\) a major motivation for using the UKF in the sequential estimation approach stems from the desire to retain these nonlinearities by foregoing Taylor series linearization of the measurement and/or dynamics models.

As a last consideration in this discussion, notice from Algorithm 1 that knowledge of the absolute orbit and attitude of the observer is required. With regard to the latter quantity, this paper assumes without loss of generality that the orientation of the VBS frame is decomposed into

\[
V_R^{I} = R_{RC} R_{RN} R_{RN} R_{RN} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & -1 & 0 
\end{bmatrix}
\begin{bmatrix}
\cos \varphi_{\text{fpa}} & \sin \varphi_{\text{fpa}} & 0 \\
-\sin \varphi_{\text{fpa}} & \cos \varphi_{\text{fpa}} & 0 \\
0 & 0 & 1 
\end{bmatrix}
R_{RN}^{I}
\]

with \( R_{RN}^{I} \) and 

\[
R_{RN}^{I} = \left[ \begin{array}{c}
\hat{r}_o \hat{v}_o \\
(\hat{r}_o \times \hat{v}_o) \times \hat{r}_o \\
\hat{r}_o \times \hat{v}_o
\end{array} \right]^T
\]

Here, the \( R' \) frame is obtained from rotating \( R \) about \( \hat{N} \) by the flight-path angle \( \varphi_{\text{fpa}} \). Note that instead of one basis vector pointing in the along-track direction, \( R' \) now has a basis vector which points along the velocity direction. In circular orbits (where \( \varphi_{\text{fpa}} = 0 \), the \( R \) and \( R' \) frames are identical (see Figure 3). The camera boresight is placed along the \( \hat{z}^V \) axis and aligned with anti-velocity direction. While these represent nominal configurations, it is important to realize that in a realistic system there will be errors in the knowledge of the observer orbit and attitude. In the subsequent simulation verification tests, knowledge errors are introduced in the observer absolute attitude, as well as on the orbit position and velocity.

### BATCH RELATIVE ORBIT DETERMINATION

This section is focused on the problem of determining a reasonable estimate of the ROE from a batch of bearing angle data. In this approach, linearized dynamics models are used to formulate the batch estimation framework. Accordingly, the state elements being estimated as specifically mean ROE. It is assumed that

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**Algorithm 1** Measurement model: Conversion from mean or osculating ROE to bearing angles

1: function \( h(\delta \omega, \omega_o, \delta R_{RN}^{I}) \) \( \triangleright \) \( \delta \omega \) and \( \omega_o \) may be mean or osculating
2: Compute \( \omega_o \) from Eq. (9) using \( \delta \omega \) and \( \omega_o \)
3: if using mean elements then
4: Convert \( \omega_o \) and \( \omega_o \) to osculating elements \( \triangleright \) analytic conversion may be identity
5: end if
6: Compute \( r_o^{I} \leftarrow \omega_o \) and \( r_o^{I} \leftarrow \omega_o \) \( \triangleright \) see\(^{16}\) or \(^{26}\)
7: Form the relative position vector in \( I \) frame: \( \delta r^{I} = r_o^{I} - r_o^{I} \)
8: Rotate relative position vector from \( I \) frame to \( V \) frame: \( \delta r^{V} = V_R^{I} \delta r^{I} \)
9: Compute bearing angles from Eq. (10)
10: end function

Clearly, Eq. (10) is only one part of the complete measurement model \( y = h(\delta \omega) \). The algorithm for transforming from mean or osculating ROE to bearing angles is highlighted in Algorithm 1. There are two particularly important components to this model that deserve attention. First, when using mean elements it is important to convert to osculating elements if possible (see Algorithm 1, Line 4). This ensures that the bearing angles (which live in an inherently “osculating” space) are matched to properly oscillating ROE trends. Schaub and Junkins\(^{26}\) and Alfriend et al.\(^{16}\) both provide an analytical model for the mean-to-osculating transformation which is nonlinear in terms of separation and linear with respect to \( J_2 \). Instead, there are limited closed-form models for other perturbation sources currently available in the literature. Second, in order to compute the relative position vector (or LOS) vector, it is first necessary to transform the absolute orbital elements to absolute inertial position and velocity components, as in Algorithm 1, Line 6. This transformation is nonlinear and retains the curvature of the trajectories inherently captured by the orbital element description. The measurement model obviously contains several distinct sources of useful nonlinearities which can be leveraged to disambiguate the weakly observable inter-spacecraft range;\(^{13}\) a major motivation for using the UKF in the sequential estimation approach stems from the desire to retain these nonlinearities by foregoing Taylor series linearization of the measurement and/or dynamics models.

As a last consideration in this discussion, notice from Algorithm 1 that knowledge of the absolute orbit and attitude of the observer is required. With regard to the latter quantity, this paper assumes without loss of generality that the orientation of the VBS frame is decomposed into

\[
V_R^{I} = R_{RC} R_{RN} R_{RN} R_{RN} = \begin{bmatrix}
1 & 0 & 0 \\
0 & 0 & 1 \\
0 & -1 & 0 
\end{bmatrix}
\begin{bmatrix}
\cos \varphi_{\text{fpa}} & \sin \varphi_{\text{fpa}} & 0 \\
-\sin \varphi_{\text{fpa}} & \cos \varphi_{\text{fpa}} & 0 \\
0 & 0 & 1 
\end{bmatrix}
R_{RN}^{I}
\]

with \( R_{RN}^{I} \) and 

\[
R_{RN}^{I} = \left[ \begin{array}{c}
\hat{r}_o \hat{v}_o \\
(\hat{r}_o \times \hat{v}_o) \times \hat{r}_o \\
\hat{r}_o \times \hat{v}_o
\end{array} \right]^T
\]

Here, the \( R' \) frame is obtained from rotating \( R \) about \( \hat{N} \) by the flight-path angle \( \varphi_{\text{fpa}} \). Note that instead of one basis vector pointing in the along-track direction, \( R' \) now has a basis vector which points along the velocity direction. In circular orbits (where \( \varphi_{\text{fpa}} = 0 \), the \( R \) and \( R' \) frames are identical (see Figure 3). The camera boresight is placed along the \( \hat{z}^V \) axis and aligned with anti-velocity direction. While these represent nominal configurations, it is important to realize that in a realistic system there will be errors in the knowledge of the observer orbit and attitude. In the subsequent simulation verification tests, knowledge errors are introduced in the observer absolute attitude, as well as on the orbit position and velocity.

**Algorithm 1** Measurement model: Conversion from mean or osculating ROE to bearing angles

1: function \( h(\delta \omega, \omega_o, \delta R_{RN}^{I}) \) \( \triangleright \) \( \delta \omega \) and \( \omega_o \) may be mean or osculating
2: Compute \( \omega_o \) from Eq. (9) using \( \delta \omega \) and \( \omega_o \)
3: if using mean elements then
4: Convert \( \omega_o \) and \( \omega_o \) to osculating elements \( \triangleright \) analytic conversion may be identity
5: end if
6: Compute \( r_o^{I} \leftarrow \omega_o \) and \( r_o^{I} \leftarrow \omega_o \) \( \triangleright \) see\(^{16}\) or \(^{26}\)
7: Form the relative position vector in \( I \) frame: \( \delta r^{I} = r_o^{I} - r_o^{I} \)
8: Rotate relative position vector from \( I \) frame to \( V \) frame: \( \delta r^{V} = V_R^{I} \delta r^{I} \)
9: Compute bearing angles from Eq. (10)
10: end function
at least one orbit of bearing angle data has been obtained along with the associated observer absolute orbit and attitude solutions. Finally, it is important to note that the following procedures can be used as a primary estimation architecture, or as means for initializing and then concurrently verifying and supplementing a sequential navigation filter. Three synergistic algorithms are presented below, and together they form the combined BROD architecture.

Coarse Range Determination from Bearing Angle Geometry

The first routine in the BROD procedure is founded on the geometric connection between the mean ROE and the relative motion trajectory features, and the associated implications on the bearing angle geometry over an orbit-sized batch. To build the intuition, consider the planar depiction in Figure 4a which shows the elevation angle geometry for a binary formation in near-circular orbits. There are two distinct and intuitive implications of the relative motion geometry on the elevation angle. First, there is mean offset of the elevation angle due to a nonzero difference in semi-major axis $\delta a$. This makes sense when recalling that $\delta a \neq 0$ implies a nonzero mean radial offset in the relative trajectory. Second, apart from any effects due to $\delta a$, the presence of a mean along-track separation $\delta \lambda$ necessarily imposes a mean offset in the elevation angle due to the curvature of the orbital path. As expected, this elevation angle offset is proportional to $\delta \lambda$. Now consider the notional 10-orbit time history of $\epsilon$ in Figure 4b, which is representative of elevation angle trends for a free-motion drifting approach trajectory. In this case, the bearing angle trend is oscillatory at the orbital natural frequency and has an amplitude which grows proportional to $(\delta a/\delta \lambda)^2$. This makes sense when considering that $\delta a \neq 0$ implies a drift in the mean along-track separation, and the perspective laws dictate that an approaching object occupies more of the field-of-view.

![Figure 4](image.png)

(a) Elevation angle offset geometry.  
(b) Notional time history of drifting elevation angle.

**Figure 4**: Depiction of the elevation angle mean offset sources (left) and oscillatory nature (right).

Now that the geometry-based intuition has been established, the claims made about the elevation angle mean offsets and oscillatory trends will now be proved analytically. The elevation angle definition from Eq. (10) is re-arranged to

$$
\tan \epsilon = \frac{\delta r^V_x}{\delta r^R_z} = \frac{\delta r^R_x}{-\delta r^R_y}
$$

(13)

where the last equality invokes the nominal orientation between $V$ and $R$ given by Eq. (12). Next, a relationship between the rectilinear RTN frame position components of Eq. (13) and the ROE must be established. Such a transformation was established by D’Amico, where it was shown that the mean quasi-nonsingular ROE as defined in Eq. (9) are first-order equivalent to the integration constants of the HCW model. Accordingly, the HCW STM was used to map between curvilinear RTN coordinates and the ROE. For completeness,
that transformation is shown here in terms of curvilinear coordinates \((\delta r_x^R, \delta r_y^R, \delta r_z^R)\)

\[
\begin{align*}
\delta r_x^R &= a_o \delta a - a_o \delta e_x \cos u_o - a_o \delta e_y \sin u_o \\
\delta r_y^R &= \delta \lambda + 2 \delta e_x \sin u_o - 2 \delta e_y \cos u_o \\
\delta r_z^R &= \delta i_x \sin u_o - \delta i_y \cos u_o
\end{align*}
\]

(14)

(15)

(16)

The rectilinear coordinates are obtained by applying the curvilinear-to-rectilinear transformation.\(^{27}\) For this analysis, it is assumed that \(|\delta \lambda| \gg 2|\delta e|\) and \(|\delta \alpha| \gg |\delta i|\). Thus, \(\delta r_x^R \approx \delta \lambda \Rightarrow \cos \delta r_y^R = 1 - \frac{1}{2} \delta \lambda^2\) and \(\cos \delta r_z^R \approx 1\) under the small-angle assumption. The rectilinear \(x\)-component is written as

\[
\delta r_x^R = (a_o + \delta r_x^R) \cos \delta r_y^R \cos \delta r_z^R - a_o \approx (a_o + \delta r_x^R) \left(1 - \frac{1}{2} \delta \lambda^2\right) - a_o \approx \delta r_x^R - \frac{1}{2} a_o \delta \lambda^2
\]

(17)

where terms proportional to \(\delta r_x^R \delta \lambda\) have been neglected since they are proportionally smaller. Substituting Eq. (14) results in

\[
\delta r_x^R \approx a_o \delta a - a_o \delta e_x \cos u_o - a_o \delta e_y \sin u_o - \frac{1}{2} a_o \delta \lambda^2
\]

(18)

Taking a similar approach for the rectilinear \(y\)-coordinate,

\[
\delta r_y^R = (a_o + \delta r_x^R) \sin \delta r_y^R \cos \delta r_z^R \approx (a_o + \delta r_x^R) \delta \lambda \approx a_o \delta \lambda
\]

(19)

where the small-angle assumption \(\sin \delta \lambda \approx \delta \lambda\) has been invoked. Combining Eqs. (18) and (19) with Eq. (13) results in the first-order bearing angle model

\[
\tan \epsilon \approx -\frac{\delta a}{\delta \lambda} - \frac{|\delta \lambda|}{2} + \frac{\delta e_x}{\delta \lambda} \cos u_o + \frac{\delta e_y}{\delta \lambda} \sin u_o
\]

(20)

Notice that the second term contains an absolute value; the elevation angle offset due to orbit curvature is always negative. To consider the "mean" elevation angle, Eq. (20) is averaged over an orbital period. The terms multiplied by \(\sin u_o\) and \(\cos u_o\) go to zero, resulting in

\[
\bar{\tan \epsilon} \approx -\frac{\delta a}{\delta \lambda} - \frac{|\delta \lambda|}{2}
\]

(21)

which is consistent with the geometry in Figure 4a (which depicts \(\delta a < 0\) and \(\delta \lambda < 0\)).

Eq. (21) reveals that the mean elevation angle offset is composed of two distinct terms, with one term proportional to the unknown range (as approximated by \(\delta \lambda\)). With a batch of elevation angles spanning at least one orbit, an empirical mean can be computed easily. This means that the range can be coarsely determined from the measurement batch if an estimate of the scaled semi-major axis difference \(\delta a/\delta \lambda\) is obtained. With that in mind, the analysis turns to the geometric intuition highlighted in Figure 4b. Consider the time rate of change of Eq. (21)

\[
\frac{d}{dt} \bar{\tan \epsilon} \approx -\frac{d}{dt} \left(\frac{\delta a}{\delta \lambda}\right) - \frac{1}{2} \frac{d|\delta \lambda|}{dt} = -\frac{3}{2} n_o \left(\frac{\delta a}{\delta \lambda}\right)^2 + \frac{3}{4} n_o \delta a \cdot \text{sign}(\delta \lambda)
\]

(22)

Here, it is assumed that \(\delta a\) is constant and that

\[
\frac{d\delta \lambda}{dt} = -\frac{3}{2} n_o \delta a
\]

(23)

is the rate of change of the relative mean longitude, given in terms of the observer mean motion \(n_o\). Since \(\delta \lambda\) is less than one in all pragmatic cases, \((\delta a/\delta \lambda)^2 >> \delta a\) and thus the second term in Eq. (22) can be reasonably neglected. The resulting approximate rate of change can be interpreted as the finite difference over timestep \(\Delta t\). If \(\delta a = 0\), the bearing angle measurements taken an orbit apart are nominally expected
to be equal. Accordingly, measuring elevation angles an orbit apart provides a means to infer the scaled semi-major axis difference. Taking this approach, $\Delta t$ is the orbital period and $n_o \Delta t = 2\pi$. Accordingly,

$$\Delta \left| \tan \epsilon \right| \approx \frac{3}{2} n_o \Delta t \left( \frac{\delta a}{\delta \lambda} \right)^2 = 3\pi \left( \frac{\delta a}{\delta \lambda} \right)^2$$

(24)

Absolute value is used to avoid taking square-roots of potentially negative numbers when re-arranging to

$$\pm \frac{\delta a}{\delta \lambda} \approx \sqrt{\frac{\Delta \left| \tan \epsilon \right|}{3\pi}}$$

(25)

To infer the sign of the scaled semi-major axis difference, it suffices to check if the elevation is increasing over the timestep. For an anti-along-track boresight alignment (i.e., $\delta \lambda < 0$), growing elevation angle implies the target and observer are drifting toward one another and $\delta a < 0$. Finally, with the scaled semi-major axis difference computed and an empirical elevation angle mean computed from the measurement batch, the range can be approximated from Eq. (21) as

$$|\delta \lambda| \approx 2 \left| \frac{\delta a}{\delta \lambda} + \tan \epsilon \right|$$

(26)

The sign is determined by checking the boresight alignment; if it is aligned in anti-along-track, $\delta \lambda < 0$. In summary, the difference between two elevation angle measurements taken an integer number of orbits apart ($\Delta \left| \tan \epsilon \right|$) is used to compute the scaled relative semi-major axis from Eq. (25). Then, using an empirically computed mean of the elevation angle history over the batch ($\overline{\tan \epsilon}$) in conjunction with the scaled semi-major axis parameter, the relative mean longitude is computed from Eq. (26). Note that the equations as they appear here have been derived using the fundamental assumptions that govern the mappings in Eqs. (14)-(16); that is, that the reference orbit is near-circular. The natural extension of this approach to eccentric orbits is an ongoing research effort which uses the modified mapping discussed in the context of Figure 2b.

Analytic Unit Vector Solution

Recall from the opening discussion that Sullivan et al.\textsuperscript{13} previously developed a method for computing the mean ROE unit vector from a batch of measurements. While the complete mean ROE state solution was left subject to a scalar magnitude ambiguity in that study, this paper now leverages that approach in conjunction with the recently-obtained coarse range estimate to determine a unique mean ROE vector. The algorithm for obtaining the unit vector solution is briefly highlighted here for completeness. At its core, the approach hinges on the simple premise that the ROE state can be decomposed into a unit vector component multiplied by a scalar magnitude

$$\delta \text{oe} = |\delta \text{oe}| \cdot \hat{\delta \text{oe}}$$

(27)

To compute the unit vector portion, the measurement model is re-arranged into the linear expression

$$H(y(t_m))\nabla R^\text{R} \mathcal{J}_{\delta \text{r}R} \mathcal{J}_{\delta \theta\text{R}} \Phi_{m,0} \delta \text{oe}(t_0) \triangleq H'\Phi_{m,0} \delta \text{oe}(t_0) = 0_{2 \times 1}$$

(28)

It is beneficial to briefly parse each of the terms in Eq. (28) to better understand the implication of this decomposition. Beginning from the rightmost term, $\delta \text{oe}(t_0)$ represents the unknown mean ROE at some epoch of interest ($t_0$) that must be estimated. The STM $\Phi_{m,0}$ propagates this ROE state from the epoch of interest to the current measurement epoch $t_m$. This ROE vector is then linearly transformed first to curvilinear coordinates through the transformation $\mathcal{J}_{\delta \theta\text{R}}$ (see for example Eqs.14 - 16). The curvilinear coordinates may then be linearly transformed to rectilinear coordinates using the Jacobian $\mathcal{J}_{\delta \text{r}R}$ of the nonlinear curvilinear-to-rectilinear coordinate transformation, linearized about some reference relative state. Note that this Jacobian reduces to the identity matrix when linearizing about a null relative state (i.e., zero separation). The rectilinear coordinates are then rotated to the $V$ frame, where they are finally multiplied by the operator $H$ to form a pseudo-measurement vector $\in \mathbf{R}^2$. 

11
The leftmost term of Eq. (28) comes from re-arranging the geometric measurement model in Eq. (10) to a linear form with respect to the $\mathcal{V}$-frame relative position components

\[
H(y(t))\delta r^\mathcal{V} = 0_{2 \times 1} \quad \text{where} \quad H(y(t)) = \begin{bmatrix} \cos \epsilon & 0 & -\sin \epsilon \\ 0 & \cos \alpha \cos \epsilon & -\sin \alpha \end{bmatrix}
\]  

(29)

It is important to note that Eq. (29) is not a linearization of Eq. (10), but rather an algebraic manipulation (see Appendix A for more details). Concatenating row-wise instances of Eq. (28) for each measurement epoch $\in \{t_1, ..., t_p\}$ results in the batch measurement equations

\[
O'\delta \bar{\omega}(t_0) = 0_{2p \times 1}
\]  

(30)

The modified observability matrix $O'$ has been introduced since it is composed of terms $H'\Phi_{m,0}$ which bear striking resemblance to the well-known standard observability matrix components. Because the originally nonlinear model was algebraically manipulated (but not linearized) to form the $H$ matrix, the modified observability matrix $O'$ is full-rank. Accordingly, $O'$ has an empty null-space and the solution to the homogeneous system of equations in Eq. (30) is $\delta \bar{\omega}(t_0) = 0$. However, due to the intrinsic observability constraints posed by angles-only measurements, $O'$ is poorly conditioned. As documented in other studies, a good approximate solution for this homogeneous system of equations comes from considering the singular value decomposition $O' = U\Sigma V^T$. Here, the column of $V$ corresponding to the minimum singular value in $\Sigma$, denoted $\nu$, represents the spanning vector in the input-space (i.e., in the $\delta \bar{\omega}$ space) that shows up the least in the output-space (spanned by columns of $U$). This means that $O'\nu$ approximates $0$ best. Since the vector $\nu$ has norm of one, it is thus treated as an estimate of the ROE unit vector $\delta \bar{\omega}$.

At this point in the developments, a method has been established for computing the mean ROE unit vector from a batch of bearing angle data. Furthermore, since this analysis has made the reasonable assumption that far-range angles-only navigation scenarios are often characterized by proportionally large mean along-track from a batch of bearing angle data. Furthermore, since this analysis has made the reasonable assumption that far-range angles-only navigation scenarios are often characterized by proportionally large mean along-track

\[
\text{Semi-Analytic Estimation using Regularized Batch Least-Squares}
\]

If further refinement of the mean ROE estimate is desired, the solution coming from the combination of the preceding two algorithms can be used to initiate an iterative batch least-squares estimation routine. This concept is highlighted now. Consider the bearing angle measurement model listed in Algorithm 1

\[
y_m = h(\delta \bar{\omega}(t_m)) \approx h(\delta \bar{\omega}(t)) + \frac{\partial h}{\partial \delta \bar{\omega}} \bigg|_{\delta \bar{\omega}(t_m)} \Phi_{m,0} (\delta \bar{\omega}(t_0) - \delta \bar{\omega}(t))
\]  

(31)

which has been expanded in a Taylor series about some reference ROE state $\delta \bar{\omega}$ and truncated to first-order. Re-arranging terms results in the compact representation

\[
\Gamma \delta \bar{\omega}(t_0) = \gamma \quad \text{where} \quad \Gamma \triangleq \frac{\partial h}{\partial \delta \bar{\omega}} \bigg|_{\delta \bar{\omega}(t_m)} \Phi_{m,0} \quad \text{and} \quad \gamma \triangleq y_m - h(\delta \bar{\omega}(t_m)) + \Gamma \delta \bar{\omega}(t_0)
\]  

(32)

(33)

where $\delta \bar{\omega}(t_m) = \Phi_{m,0} \delta \bar{\omega}(t_0)$. The iterative least-squares problem is thus posed to find $\delta \bar{\omega}(t_0)$ which minimizes the cost function

\[
C = ||\Gamma \delta \bar{\omega}(t_0) - \gamma||^2 + \kappa (\Gamma^T \Gamma)^{1/2} (\delta \bar{\omega}(t_0) - \delta \bar{\omega}(t_0))||^2
\]  

(34)
where the leftmost term captures the sensing residuals and the rightmost term enforces that the solution does not depart far from the linearization references state. The latter is often referred to as the regularization term, with $\kappa$ scaling the “region of trust” around $\delta \bar{\alpha}$. The batch measurement Jacobian $\Gamma$ is used in the regularization term to penalize solutions which deviate in directions where the cost is varying rapidly. Similar to the non-regularized least-squares method, the solution which minimizes $C$ comes from a modified pseudo-inverse of $\Gamma$:

$$\delta \bar{\alpha}(t_0) = (1 + \kappa) \left( \Gamma^T \Gamma \right)^{-1} (\Gamma^T \gamma + \kappa \Gamma^T \Gamma \delta \bar{\alpha}(t_0))$$  \hspace{1cm} (35)

It is important to note that $\Gamma$ and $\gamma$ in Eq. (35) are interpreted as concatenated forms of the terms in Eq. (33) over the entire measurement batch. The least-squares solution is kick-started by setting $\delta \bar{\alpha}(t_0)$ equal to the approximate ROE solution coming from the previously-established deterministic method. Iterative refinement of the solution can be conducted by setting $\delta \bar{\alpha}(t_0)$ equal to the ensuing estimate of $\delta \bar{\alpha}(t_0)$ at the end of each iteration. The use of the deterministic ROE solution effectively reduces the search space over which the iteration occurs by improving the measurement model reference/linearization state. The measurement model Jacobian $\Gamma$ may be computed analytically as a decomposition similar to Eq. (28), or numerically using a finite difference method. In the latter, the use of the linearized dynamics updates via an STM makes this approach “semi-analytic” in the sense that it blends accurate closed-form knowledge of the dynamics with numerical approximation of the measurement model. This features yields more computational efficiency than a traditional fully nonlinear batch nonlinear least squares (or other numerical optimization) method. Finally, just as with the unit vector method, the least-squares refinement step is modular with respect to dynamics modeling since it simply requires that the relevant STM be computed in the solution.

SEQUENTIAL RELATIVE ORBIT ESTIMATION

Unscented Kalman Filter Dynamics Modeling

When a new measurement set is received, the first step in the general Kalman filtering architecture is to propagate the current state and covariance estimates forward to the measurement time. The explicit assumption within this framework is that the state dynamics are Gaussian, with the distribution mean capturing the current state estimate and the distribution covariance capturing the current state uncertainty. While the state propagation for a nonlinear system may proceed directly through the nonlinear dynamics, the uncertainty must be updated linearly to retain a Gaussian distribution. Rather than computing a first-order Taylor series expansions of the dynamics model, the UKF enables higher-order nonlinearities to be captured by using a stochastic weighted linear regression. With this insight, a navigation filter is designed which updates the state and covariance via numerical integration of the nonlinear osculating ROE dynamics given by the GVE in Eq. (8). In conjunction with the nonlinear measurement model, this filter class (denoted hereafter as the numerical filter) mitigates the inherent observability problems in the angles-only navigation scenario by retaining key higher-order dynamical system features which potentially uniquely define the relative state trajectory. As an alternative to using numerical integration within the estimator, this work poses a second filtering class (denoted as the analytic filter) which uses an STM to linearly update a mean ROE estimated state and covariance. As explored in previous work, this method still retains some observability-improving nonlinear features which are exploited through the measurement relationship between mean ROE and the bearing angles. The choice between a numerical or an analytic filter implementation is a strategic design trade-off which is quantified in the upcoming algorithm verification sections. Figure 5 graphically compares the two filter variants with a sampling-based example, where $\chi$ denote the state “sigma points” that are propagated through the nonlinear dynamics in the UKF and used to compute an updated empirical state covariance.

The effectiveness of either filter class is driven by its ability to properly fuse the dynamical evolution of the estimation state with measured trends of the bearing angles. Accordingly, it is essential to capture the effects in the orbit environment which drive substantial variation in the ROE over time. In this paper, the two primary orbit regimes of interest are LEO and GEO since they are highly relevant to spacecraft situational awareness and proximity relative navigation. To qualify the critical dynamic effects in these environments which must be captured by the filter, a numerical perturbation simulation is undertaken using the aforementioned high-fidelity numerical propagator. Informed by the analytic insights of Koenig et al. and Guffanti et al.
outputs of this qualitative study illustrate the secular, long-periodic, and short-periodic osculating ROE trends induced by the dominant Earth oblateness ($J_2$) perturbation in LEO and SRP in GEO. As shown in Figure 6a, $J_2$ causes a drift in $\delta \lambda$ and a circular precession of the relative eccentricity vector $\delta e$. In addition to these secular and long-periodic variations, the osculating ROE display short-period oscillations with amplitudes that grow with increasing inter-spacecraft separation (i.e., increasing $|\delta \lambda|$). Note that the effects of $J_2$ on $\delta i$ are not shown in this figure. In GEO scenarios, the effect of SRP on the observer-target system is largely driven by their difference in ballistic coefficients

$$\Delta B \equiv B_t - B_o$$

where $B \equiv C_r \frac{A}{m}$

In this definition $C_r$ is the reflectivity coefficient and $A/m$ is the area-to-mass ratio. Figure 6b highlights the ROE evolutions due to SRP for a $\Delta B$ of 20% and 80%. In particular, drift and short-periodic oscillations in $\delta \lambda$ are accompanied by short-periodic oscillations in $\delta a$ and a long-periodic circular precession of $\delta e$. The magnitude of each of these effects scales with $\Delta B$. SRP does not generally produce substantial variation in $\delta i$. The secular and long-period trends witnessed in this analysis are fully consistent with previous works.\textsuperscript{14,22}

It is interesting to note that the role of $J_2$ in LEO and the role of SRP in GEO are qualitatively analogous and equally represent dynamical features which can be exploited by the filter. In the context of modeling SRP perturbations in the filter, it is evident that knowledge of $\Delta B$ is required. Accordingly, this parameter may be treated as a known quantity (perhaps with some uncertainty associated with it) or as an element which must be appended to the state vector and estimated along with the ROE.

### Impulsive Maneuver Inclusion

In scenarios where impulsive orbit reconfiguration is conducted, the angles-only navigation filter must assimilate the maneuver execution into the propagation step. Since there are several noteworthy research studies that highlight how the ROE state evolves subject to impulsive maneuvers,\textsuperscript{29,30} the main focus of this section is on accounting for the maneuver when updating the state covariance. While the analytic and numerical filters fundamentally handle covariance propagation in different ways, the inclusion of uncertainty in the $\Delta v$ (i.e., change in velocity) from the executed maneuver can be accounted for in a harmonized way.
(a) Effects of $J_2$ in LEO over 30 days.

(b) Effects of SRP in GEO over 60 days. $\Delta B = 20\%$ (dark) and $\Delta B = 80\%$ (light) are both shown.

Figure 6: Mean (dashed) and osculating (gray) perturbed ROE trends w/ initial (△) and final conditions (□).

This work takes inspiration from an approach by Schiff	extsuperscript{31} which proposes systematically increasing the filter process noise covariance, $Q \in \mathbb{R}^{n \times n}$, to account for maneuver execution uncertainty.

Consider a re-arranged form of the linear propagation equations in Eq. (3) given by

$$x(t_k) - \Phi_{k,j}x(t_j) = \Upsilon_{k,j}u(t_j) \quad (37)$$

Notice that the left term contains a term which propagates the state through the free dynamics to account for the additional natural state evolution over the interval $t_k - t_j$. Accordingly, it can be interpreted as the net change in state vector ($\Delta x$) induced by the control input $u$ executed at time $t_j$. The impulsive control input is represented by a vector of $\Delta v$ components executed by the observer and described in the observers RTN frame $\Delta v \triangleq (\Delta v_R, \Delta v_T, \Delta v_N)^T$. The above equation is now re-written as

$$\Delta \delta\omega(t_k) = \Upsilon_{k,j} \Delta v(t_j) \quad (38)$$

The uncertainty in maneuver execution can be reasonably formulated in terms of uncertainty in the individual $\Delta v$ components. In particular, by treating $\Delta v$ as a Gaussian random variable with uncorrelated variances $\sigma^2_{\Delta v(i)}$, the covariance matrix associated with the control input vector is given by the diagonal matrix

$$P_{\Delta v} \triangleq \text{diag} \left( \sigma^2_{\Delta v_R}, \sigma^2_{\Delta v_T}, \sigma^2_{\Delta v_N} \right) \quad (39)$$

Since $\Delta v$ is a product of the spacecraft thruster, benchtesting can be conducted to sufficiently estimate $P_{\Delta v}$.

The maneuver covariance matrix can now be mapped to a $\Delta \delta\omega$ covariance matrix through the linear system in Eq. (38). The uncertainty of the maneuver-induced state evolution represents a reasonable metric by which to systematically increase the filter process noise covariance. Accordingly,

$$\Delta Q \triangleq P_{\Delta \delta\omega} = \text{Cov}[\Upsilon \Delta v] = \Upsilon P_{\Delta v} \Upsilon^T \quad (40)$$

is added to the state covariance propagated through the angles-only navigation filter. This ultimately serves the purpose of scaling the state covariance to account for uncertainty in the executed maneuver, and keeps the filter receptive to new measurements while the state is evolving during the reconfiguration. It is important to note that this work leverages the various CSM ($\Upsilon$) formulations derived from the GVE and tailored to near-circular and eccentric orbits by Chernick et al.	extsuperscript{30}
Table 2: Sources of noise injected during algorithm verification.

<table>
<thead>
<tr>
<th>Noise source</th>
<th>1-σ</th>
<th>Units</th>
</tr>
</thead>
<tbody>
<tr>
<td>Observer absolute position</td>
<td>20</td>
<td>m</td>
</tr>
<tr>
<td>Observer absolute velocity</td>
<td>0.25</td>
<td>m/s</td>
</tr>
<tr>
<td>Observer boresight attitude</td>
<td>30</td>
<td>arcsec</td>
</tr>
<tr>
<td>Observer off-axis attitude</td>
<td>6</td>
<td>arcsec</td>
</tr>
<tr>
<td>$(\alpha, \epsilon)$ measurement</td>
<td>30</td>
<td>arcsec</td>
</tr>
</tbody>
</table>

HIGH-FIDELITY ALGORITHM VERIFICATION

The primary purpose of this section is verify the proposed batch and sequential relative orbit estimation architectures in high-fidelity. To that end, a simulation pipeline is established which makes use of a rigorous full-force orbit propagator to compute the trajectories of the observer and target subject to a multitude of perturbations (see Table 1). The truth-side data comes from numerically integrating the GVE, and the resulting trajectories are then used to generate measurements for the algorithms in one of two ways. First, they are used as inputs into a synthetic measurement emulation model which delivers bearing angles, observer absolute orbit, and observer absolute attitude inputs which have been corrupted with noise that is consistent with commercially available sensors. Table 2 captures the various noise sources that have been injected into the simulation. In particular, the observer orbit knowledge errors are conservative values for a Position, Velocity, and Time (PVT) solution using the Global Navigation Satellite System (GNSS), the attitude knowledge errors are in line with the Blue Canyon Technologies attitude determination system capability, and the bearing angle noise value is consistent with results from several sensor benchtests. With regard to the latter parameter, 30 arcsec is on the order of one-third to one-half a pixel for several candidate VBS. Second, the trajectories are used as inputs into a hardware-in-the-loop optical testbed\cite{32,33} in order to stimulate a space-grade camera with dynamically rendered scenes of the observer and the background starfield. The testbed delivers measurements of the bearing angles and the observers absolute attitude to the angles-only navigation algorithms. Accordingly, only the observer absolute orbit knowledge is generated via simulation and corrupted with noise per Table 2.

Verification of the algorithms is conducted using scenarios which are composed of an observer absolute orbit configuration and a relative motion configuration about that observer orbit. In this work, the observer orbit is either a near-circular sun-synchronous LEO (denoted $LEO1$ hereafter), or a very near-equatorial geosynchronous orbit (denoted $GEO1$ hereafter), and is specified with a set of absolute mean orbital elements. The motion of target about the observer’s orbit is instead parameterized using a set of mean ROE, and three relative orbit (RO) scenarios are configured for this analysis (denoted $RO1$ through $RO3$, hereafter). The $RO1$ case represents a standard far-range hold configuration where the target is displaced -30 km in the observer’s along-track direction, and the relative motion projected in the radial/cross-track plane traces out a circle of 500 m radius. $RO2$ is nearly identical, except that there is mean semi-major axis offset which induces a drifting approach that brings the observer and target closer at a rate of approximately 1.5 km per orbit. Finally, the $RO3$ scenario is a non-drifting trajectory with 100 km mean along-track separation and 1000 m

Table 3: Initial conditions for batch and/or sequential algorithm verification scenarios.

<table>
<thead>
<tr>
<th>Observer</th>
<th>$a$ (km)</th>
<th>$e_x$ (-)</th>
<th>$e_y$ (-)</th>
<th>$i$ (deg)</th>
<th>$\Omega$ (deg)</th>
<th>$\omega_0$ (deg)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$LEO1$: sun-synch., LTAN 11</td>
<td>7207.20</td>
<td>-5.00 × 10^{-4}</td>
<td>8.66 × 10^{-4}</td>
<td>98.70</td>
<td>28.87</td>
<td>300.00</td>
</tr>
<tr>
<td>$GEO1$: geosynchronous</td>
<td>42164.00</td>
<td>-5.00 × 10^{-4}</td>
<td>8.66 × 10^{-4}</td>
<td>1.00</td>
<td>60.00</td>
<td>300.00</td>
</tr>
<tr>
<td><strong>Relative Orbit (RO)</strong></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>$RO1$: mid-range hold</td>
<td>0</td>
<td>-30 × 10^3</td>
<td>500</td>
<td>0</td>
<td>-500</td>
<td>0</td>
</tr>
<tr>
<td>$RO2$: mid-range approach</td>
<td>-150</td>
<td>-30 × 10^3</td>
<td>500</td>
<td>0</td>
<td>-500</td>
<td>0</td>
</tr>
<tr>
<td>$RO3$: far-range hold</td>
<td>0</td>
<td>-100 × 10^3</td>
<td>1000</td>
<td>0</td>
<td>-1000</td>
<td>0</td>
</tr>
</tbody>
</table>
projected circular motion in the radial/cross-track plane. These six different configurations highlighted in Table 3 represent reasonable on-orbit space situational awareness applications for angles-only navigation.

During the algorithm tests, measurements of the bearing angles are delivered synchronously with the absolute observer orbit and attitude solutions. There is no error injected into the measurement time-tagging. Unless otherwise specified in specific instances, measurements are received at an interval of 120 sec. The numerical filter uses a 60 sec time-step for the internal integration of the GVE, with force modeling to include a $5 \times 5$ Earth gravity model, lunisolar third-body gravitation, and SRP (in GEO cases). When modeling the SRP perturbation, this work assumes that the differential ballistic coefficient is known a priori and is equal to the relatively large value of 0.80 (i.e., there is an 80% difference between target and observer SRP ballistic properties). For reference, the difference in ballistic coefficients between the Mango and Tango spacecraft in the PRISMA mission\textsuperscript{23} was just over 25%.

**Batch Relative Orbit Determination**

This section highlights the numerical verification results for the BROD algorithm, where an individual focus has been placed on both the geometric range estimation method and the iterative refinement procedure. Note that the verification of the unit vector solution has been documented in previous work\textsuperscript{13} and is not repeated here. Consider first the estimation of $a\delta\lambda$ for the LEO1 class of scenarios. Since the geometric method relies heavily on trends in the bearing angle data batch, the solution sensitivity to noise is of particular interest. Figure 7 provides plots of the $a\delta\lambda$ errors as a function of the elevation angle noise 1-$\sigma$ standard deviation. Looking first at subfigure 7a, which contains the results for RO1 and RO3, there are several key takeaways. First, the estimation of $a\delta\lambda$ occurs with reasonable accuracy. In particular, RO1 shows errors of approximate 7% of the true value, and RO3 displays errors of about 1.3% of the true value. While the fact that $a\delta\lambda$ is estimated with better accuracy for RO3 (a larger separation case) than RO1 may initially be counter-intuitive, it makes more sense when considering the fundamental way in which the geometric method works. For non-drifting type cases like these, the sole means for computing $a\delta\lambda$ comes from the effect of the orbit curvature (see Eq. (21) with $\delta a = 0$) which is more pronounced for larger separations. Accordingly, the RO3 measurements of angular offset are proportionately less corrupted by noise than the RO1 measurements, and the estimation of $a\delta\lambda$ tends to be more accurate. Finally, the estimation errors generally tend to be rather steady with respect to the angle measurement noise, with both cases showing variations with magnitudes that are well within 1% of the true value. Instead, the results shown in subfigure 7b correspond to the drifting RO2 case. Recall that, in this case, $a\delta\lambda$ is estimated using the curvature effect as well as the $\delta a$ effect on the mean elevation angle offset. The latter term is provided only through a coarse approximation, and the results indicate a corresponding sensitivity and variability in the solution accuracy. Still, the errors remain within about 25% of the true value throughout. It would stand to reason then that estimates for RO2-type trajectories could benefit from iterative refinement using the least-squares approach.

In addition to the difficulties coming from the additional drift-related term in the geometric method, another issue posed by the approach comes from the fact that no knowledge of the dynamics is being used in the estimation. While this didn’t preclude reasonable accuracy in the LEO1 cases, it becomes a substantial issue
in GEO1 test cases which are dominated by the SRP effect due to a large difference in ballistic coefficients. In general, this makes GEO1-type orbits obvious candidates to use the iterative refinement method with the geometric solution as a coarse initializer. An illustration to this effect is shown in Figure 8 for the GEO1 RO3 test case. Recall that the non-drifting RO3 trajectory was very well-handled in the LEO1 case. Instead, looking to the Iteration 0 value in Figure 8, it is clear that the geometric method alone results in aδλ error of over 500 km. However, by infusing knowledge of the dynamics into the estimation process through the iterative least-squares minimization, the subsequent results drive down to errors that are within 10% for the plotted case in just seven iterations. The remaining BROD test case results, including both the geometric and subsequent iterative refinement application, are tabulated in Table 4. As expected, the use of least-squares minimization only marginally improves the results for LEO1 RO1 and RO3, while improving the geometric estimates by a factor of four in the LEO1 RO2 case and by two orders of magnitude for the GEO1 cases.

Sequential Relative Orbit Estimation

Moving on now to the numerical verification results for the sequential estimators, this discussion begins with maneuver-free test cases. Simulations spanning four observer orbits are conducted for each scenario tabulated in Table 3 using both the analytic and numerical forms of the angles-only UKF. "Steady-state" estimation error statistics are computed over the last two simulated orbits. In the first tests cases, the filters are initialized with 25% errors in aδλ and the unit vector solution is used to provide an initialization for the remaining ROE components using the relative mean longitude as a scalar. While it has been established that using a combination of the geometric and least-squares refinement leads to ROE estimates that are well-within 25%, these first tests provide reasonable insight into how the filters respond when given fairly poor initial conditions. Figure 9 contains the estimation errors for aδλ for both filter variants in each of the three LEO1 scenarios. It is important to re-iterate that, since the ROE state inherently decouples the weakly observable aδλ term from the remaining elements, the estimation errors of all non-zero ROE tend to scale with the error in aδλ. In this way, the results of Figure 9 are a succinct indicator of filter performance for all ROE. There are several immediate trends to note from the plotted results. First, the maneuver-free filters are stable and
errors using analytic filter.

(a) $a\delta\lambda$ errors using analytic filter.

(b) $a\delta\lambda$ errors using numerical filter.

Figure 9: Analytic and numerical filters estimation errors and 3-$\sigma$ standard deviation bounds for the LEO1 RO1 (top), RO2 (middle), and RO3 (bottom) cases. Filters are initialized with 25% error in $a\delta\lambda$ and 30% 1-$\sigma$ error standard deviation. Eclipses (no measurements) are highlighted in gray.

converge to steady-state well within the four orbit simulation time while subject to measurement outages for 30% of each orbit due to eclipse. In fact, the convergence phase of the filters starts after about one orbit of sequential data is processed. This is a strong indication that the orbit dynamics are being properly sampled and that returning to similar (but orbit period separated) vantage points enables better range disambiguation. Such a trend is not dissimilar from the results seen in the Simultaneous Localization and Mapping (SLAM) problem, where returning to a previously-mapped feature improves estimation accuracy and reduces error covariance. Second, while both filter variants are quite successful at estimating the relative mean longitude, the numerical filters (see Figure 9b) achieve about a factor of two better accuracy at steady-state than the analytic filters (see Figure 9a) for the non-drifting cases, and comparable accuracy for the drifting RO2 case. Overall, the results provide a strong indication of filter efficacy, with the analytic and numerical filters reducing estimation errors from 25% to within 3.0% and 1.5% of their corresponding true values, respectively.

It is worth noting that this work does not consider filter sensitivity to uncertainty in the absolute orbit and attitude knowledge; the former is fully expected to affect the numerical filter substantially more than the analytic filter due to the highly nonlinear coupling in the filter dynamics model. A full sensitivity analysis of these filters is a major topic of future work.

Looking to the GEO1 test cases, the main open question to be explored is how the SRP perturbation affects the sequential filter performance. Just as in the BROD case, where it was evident that infusing knowledge of the relative dynamics was critical for improved estimation, the dynamics modeling in the sequential filter is a strategic design parameter. Accordingly, a comparative assessment of the analytic and numerical filters
is conducted where the effects of SRP are considered by accounting for or omitting it from the "onboard" modeling. In the analytic filters, the STM developed by Guffanti\textsuperscript{22} is used to capture the mean perturbation trends; in the numerical filters, a flat plate model with analytic sun ephemerides and canonical Earth shadowing is implemented to account for the osculating perturbation effects. In both cases, it is assumed that the differential ballistic coefficient is known a priori with no error. The results of this investigation are shown for GEO1 RO1 in Figure 10, where Figure 10a illustrates the analytic filter performance and Figure 10b captures the numerical filter outputs. Similar to the BROD scenarios, the test results here strongly indicate the necessity of modeling the SRP perturbation for filter accuracy. Both filter variants display a large but convergent steady-state error and covariance in the case where SRP is omitted. The convergence to a strongly biased estimate is a strong indicator of dynamics mismodeling in the filter leading to new measurements being associated to improper estimates of range as the geometry in Figure 4 would suggest. Instead, including SRP in the filter dynamics model is shown to stabilize the filter around much more accurate estimates. Interestingly, due to the fact that the mean dynamics model in the analytic filter does not immediately account for the short-period oscillations seen in $a\delta\lambda$ (as seen in Figure 6b), the estimation error in Figure 10a is oscillatory about a mean of approximately 2.3 km. The numerical filter does not display this oscillatory response since it is intrinsically retaining the short-period trends by numerically integrating the osculating equations of motion. Note that these results are generated using an initialization error of 15\% on the full ROE state.

As an overall test for the complete set of maneuver-free angles-only algorithms, the previously discussed BROD tests are used to initialize the same analytic and numerical sequential filters. For these cases, the initial state covariance is provided by using a weighted least-squares approach with weight matrix given by the measurement covariance. The results for these tests are summarized in Table 4, where again the geometric and least-squares BROD results are shown and then provided to both the analytic filter ("A. Filter") and numerical filter ("N. Filter") variants. The LEO1 test results indicate filter performance that is on par with or better than the results using more severe initialization error (see Figure 9). In particular, the more favorable initialization clearly allows for numerical filter accuracy at a fraction of a percent. For the GEO1 cases, the numerical filter far outperforms the analytic counterpart despite the relatively poorer BROD performance (and correspondingly worse initialization error) as compared with the LEO1 cases. This suggests that the analytic filters are generally more sensitive to initialization errors than the numerical counterparts. Finally, in all results cases the computed standard deviations are generally of the same order of magnitude as the mean errors and indicate reasonable steady-state convergence.

While the discussion up to this point has largely focused on maneuver-free scenarios, the next results are targeted toward understanding the effect of a single impulsive maneuver on the estimation stability and accuracy. As previously mentioned, this is a topic that has received substantial focus in the existing literature. Flight experience from the ARGON\textsuperscript{9} and AVANTI\textsuperscript{34} experiments showed that maneuvering for range reconciliation in the angles-only scenario generally needed to be periodically repeated to correct for filter drift. This is ultimately because the filter formulation using the Extended Kalman Filter (EKF) did not provide sufficient fundamental observability using dynamics or measurement nonlinearities. Accordingly, it is of in-

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure10}
\caption{Comparison of analytic and numerical filter performances for GEO1 RO1. Error and 3-\(\sigma\) bounds shown when using (solid) and omitting (dashed) SRP modeling in dynamics propagation.}
\label{fig:comparison}
\end{figure}
Table 4: Verification errors for BROD geometric (Geom.) and least-squares (L.-S.) methods, and for the sequential analytic (A. Filter) and numerical (N. Filter) filter variants. Filter mean ± 1-σ standard deviation errors are computed over last two orbits. Filters are initialized with BROD.

<table>
<thead>
<tr>
<th>Method</th>
<th>∆a (m)</th>
<th>∆λ (m)</th>
<th>∆e_x (m)</th>
<th>∆e_y (m)</th>
<th>∆i_x (m)</th>
<th>∆i_y (m)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Geom.</td>
<td>5.5</td>
<td>2110.8</td>
<td>35.4</td>
<td>2.6</td>
<td>34.9</td>
<td>1.2</td>
</tr>
<tr>
<td>L.-S.</td>
<td>2.4</td>
<td>1490.7</td>
<td>24.9</td>
<td>0.8</td>
<td>24.7</td>
<td>1.1</td>
</tr>
<tr>
<td>A. Filter</td>
<td>0.5 ± 0.6</td>
<td>217.1 ± 247.0</td>
<td>3.3 ± 3.9</td>
<td>0.2 ± 0.5</td>
<td>2.7 ± 3.8</td>
<td>5.9 ± 0.2</td>
</tr>
<tr>
<td>N. Filter</td>
<td>0.2 ± 0.7</td>
<td>5.7 ± 295.4</td>
<td>0.1 ± 4.8</td>
<td>0.4 ± 0.5</td>
<td>0.0 ± 4.7</td>
<td>0.1 ± 0.1</td>
</tr>
<tr>
<td>LEO1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Geom.</td>
<td>43.6</td>
<td>8775.1</td>
<td>182.4</td>
<td>2.8</td>
<td>180.1</td>
<td>0.4</td>
</tr>
<tr>
<td>L.-S.</td>
<td>7.3</td>
<td>2035.9</td>
<td>34.0</td>
<td>0.7</td>
<td>33.9</td>
<td>1.1</td>
</tr>
<tr>
<td>A. Filter</td>
<td>1.6 ± 0.7</td>
<td>292.4 ± 124.9</td>
<td>5.5 ± 2.1</td>
<td>0.2 ± 0.5</td>
<td>6.5 ± 2.0</td>
<td>5.5 ± 0.2</td>
</tr>
<tr>
<td>N. Filter</td>
<td>0.2 ± 0.6</td>
<td>113.3 ± 277.7</td>
<td>2.0 ± 2.4</td>
<td>0.3 ± 0.4</td>
<td>2.0 ± 2.3</td>
<td>0.1 ± 0.1</td>
</tr>
<tr>
<td>RO1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Geom.</td>
<td>9.8</td>
<td>1516.7</td>
<td>14.8</td>
<td>1.9</td>
<td>13.0</td>
<td>5.9</td>
</tr>
<tr>
<td>L.-S.</td>
<td>8.2</td>
<td>1331.1</td>
<td>13.2</td>
<td>1.5</td>
<td>12.9</td>
<td>3.4</td>
</tr>
<tr>
<td>A. Filter</td>
<td>6.7 ± 6.1</td>
<td>1101.5 ± 648.8</td>
<td>10.5 ± 8.9</td>
<td>0.1 ± 1.7</td>
<td>8.3 ± 8.6</td>
<td>19.6 ± 0.8</td>
</tr>
<tr>
<td>N. Filter</td>
<td>2.1 ± 5.0</td>
<td>444.7 ± 678.4</td>
<td>4.6 ± 7.1</td>
<td>0.8 ± 1.6</td>
<td>4.2 ± 6.8</td>
<td>0.2 ± 0.2</td>
</tr>
<tr>
<td>GE01</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Geom.</td>
<td>1286.2</td>
<td>194876.8</td>
<td>6736.6</td>
<td>2028.6</td>
<td>8662.1</td>
<td>4.0</td>
</tr>
<tr>
<td>L.-S.</td>
<td>165</td>
<td>6870.5</td>
<td>162.3</td>
<td>19.1</td>
<td>161.9</td>
<td>0.3</td>
</tr>
<tr>
<td>A. Filter</td>
<td>0.2 ± 19.0</td>
<td>2425.2 ± 1100.0</td>
<td>55.7 ± 19.9</td>
<td>30.8 ± 13.1</td>
<td>40.7 ± 23.3</td>
<td>0.4 ± 2.9</td>
</tr>
<tr>
<td>N. Filter</td>
<td>0.0 ± 0.1</td>
<td>32.5 ± 17.8</td>
<td>0.4 ± 0.3</td>
<td>0.2 ± 0.1</td>
<td>0.5 ± 0.3</td>
<td>0.1 ± 0.0</td>
</tr>
<tr>
<td>RO2</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Geom.</td>
<td>1286.2</td>
<td>194876.8</td>
<td>6736.6</td>
<td>2028.6</td>
<td>8662.1</td>
<td>4.0</td>
</tr>
<tr>
<td>L.-S.</td>
<td>165</td>
<td>6870.5</td>
<td>162.3</td>
<td>19.1</td>
<td>161.9</td>
<td>0.3</td>
</tr>
<tr>
<td>A. Filter</td>
<td>0.2 ± 19.0</td>
<td>2425.2 ± 1100.0</td>
<td>55.7 ± 19.9</td>
<td>30.8 ± 13.1</td>
<td>40.7 ± 23.3</td>
<td>0.4 ± 2.9</td>
</tr>
<tr>
<td>N. Filter</td>
<td>0.0 ± 0.1</td>
<td>32.5 ± 17.8</td>
<td>0.4 ± 0.3</td>
<td>0.2 ± 0.1</td>
<td>0.5 ± 0.3</td>
<td>0.1 ± 0.0</td>
</tr>
<tr>
<td>RO3</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Geom.</td>
<td>43.1</td>
<td>4096.3</td>
<td>72.5</td>
<td>34.1</td>
<td>73.3</td>
<td>1.3</td>
</tr>
<tr>
<td>L.-S.</td>
<td>10.1</td>
<td>9844.2</td>
<td>102.6</td>
<td>30.9</td>
<td>103.9</td>
<td>0.6</td>
</tr>
<tr>
<td>A. Filter</td>
<td>2.8 ± 21.3</td>
<td>2919.2 ± 1433.1</td>
<td>10.9 ± 13.9</td>
<td>8.1 ± 2.9</td>
<td>37.6 ± 15.5</td>
<td>0.6 ± 3.0</td>
</tr>
<tr>
<td>N. Filter</td>
<td>0.2 ± 0.1</td>
<td>168.0 ± 84.9</td>
<td>1.3 ± 0.8</td>
<td>0.3 ± 0.1</td>
<td>1.5 ± 0.8</td>
<td>0.2 ± 0.0</td>
</tr>
</tbody>
</table>

It is interesting to compare the analytic UKF developed in the context of this paper with an analytic EKF using the same settings and the same single impulse maneuver profile. As in previous test cases, four observer orbits are simulated. However, now an observer maneuver is commanded with the appropriate time and magnitude to change the magnitude of the relative inclination vector (but not the phase) by 250 m. This is intended to provide sufficient variation in the relative trajectory for range disambiguation from a new vantage point, while also being a relatively simple analytic maneuver profile to compute.

Figure 11 contains the estimation mean errors and 3-σ standard deviations for aδλ output from the filters during this brief experiment. Including a maneuver in the UKF (see Figure 11a) immediately begins a strong convergence phase that yields tight estimation accuracy to within 1% of the true relative mean longitude. Instead, the same maneuver has a much less pronounced effect on both estimation mean convergence and covariance shrinking when compared with the UKF results. Furthermore, the EKF estimate is drifting at the conclusion of the maneuver. From both of these qualitative results, it stands to reason that several maneuvers should be conducted to lower the estimation mean error and to re-stabilize the filter after excessive drift as occurred.

Hardware-in-the-Loop Filter Tests

As a final set of verification cases, the sequential filter algorithms are tested with a representative VBS in the loop. For more detailed information regarding the design of the Optical Stimulator (OS) testbed, the reader is referred to the work of Beierle et al. In summary, the OS testbed consists of a Light Emitting Diode (LED) monitor which dynamically renders stellar and non-stellar objects that are then collimated through an optical chain prior to being received by the VBS. The OS is able to stimulate point sources of light to within ten arcseconds over eight orders of radiometric accuracy. For the simulated cases, the OS uses
the true orbit trajectories of the observer and target and the observers true attitude to generate the scene. An example of the OS output seen by the camera is given in Figure 13b in Appendix B, where stellar objects are highlighted in yellow and are used to calculate an "onboard" attitude solution using a weighted centroiding image processing pipeline and a known star catalog. The attitude solution and the bearing angles describing the target angular location (see green box in Figure 13b) are provided to the sequential filtering algorithm at each measurement step. For the OS-based tests, measurements are received at 300 sec intervals instead of 120 sec as in previous tests. Figure 12a shows results from the numerical filter over approximately one simulated orbit for the LEO1 RO1 case with a single impulsive maneuver conducted to change the relative inclination vector magnitude by 250 m. As in previously shown maneuvering cases, the estimation error for $a\delta \lambda$ immediately starts converging after the known impulse is accounted for in the propagation. This test demonstrates a first-of-its-kind closed-loop simulation capability where maneuvers are commanded and artificial scenes are adaptively generated to stimulate the sensor and algorithms. Results for GEO1 RO1 case are shown in Figure 12b, where the numerical filter has been used with a model for SRP included. The estimation accuracy obtained from these tests are comparable with previously highlighted results that did not contain hardware-in-the-loop.

CONCLUSIONS AND WAY FORWARD

This paper has addressed the problem of developing estimation algorithms that are used to determine the motion of a target space object with respect to an observing spacecraft using angles-only measurements. A major component of this work focused on the development of a batch method which uses bearing angles and observer orbit and attitude taken over at least one orbital period to estimate a set of mean relative orbital elements (ROE) describing the target relative state. The synergistic algorithm used simple geometric insights gained from the oscillatory nature of the bearing angles over the batch to estimate the range, and infused knowledge of the relative dynamics in an iterative least-squares refinement procedure. Since the method provides an estimate of the full relative state, it has the potential to be used as a primary estimator where batch methods are acceptable, as an initialization for a sequential estimator, and/or as a simultaneous estimation method to be used for data editing and sequential filter health monitoring. As a counterpart to the batch...
method, this paper considered two sequential filter variants based on the Unscented Kalman Filter (UKF). The choice of UKF was largely motivated by its ability to retain higher-order characteristics in the dynamics and measurements. The two filter variants differed in their method of propagating the state dynamics; one made use of efficient linear propagation with a state transition matrix, while the second used numerical integration of the Gauss Variational Equations to provide better accuracy but at a higher computational burden. The sequential filters were compared in rigorous numerical simulation for maneuver-free and maneuver-including cases. Finally, a brief set of experiments were conducted using hardware-in-the-loop to continue preparing and assessing the algorithms for embedding on real flight systems.

There were several key lessons learned from the numerical simulation tests of each algorithm. First, in the batch relative orbit determination cases, it was shown that for a subset of low Earth orbit configurations considered (those with no relative drift), the simple geometric method could be used to provide estimates of the ROE with reasonable accuracy. Despite the advantage of simplicity and determinism, the geometric method fell short with drifting relative trajectories and in geostationary Earth orbits (GEO) which were highly perturbed by solar radiation pressure. In these cases, however, the iterative least-squares refinement algorithm successfully reduced estimation errors down to well-within acceptable values for use as an initialization to the sequential filters. With regard to the UKF algorithms, simulation results generally showed that the numerical filter is able to outperform the analytic filter by capturing nonlinear osculating characteristics of the relative orbital dynamics. However, both categories of UKF broke down substantially in GEO when solar radiation pressure was not modeled. Apart from that, all filter tests indicate very strong estimation of the ROE state without conducting any orbital maneuvers for range disambiguation (a common strategy in the literature). Furthermore, including a single impulsive maneuver with the UKF was shown to drastically hasten convergence speed and improve estimation accuracy. A comparison with an Extended Kalman Filter showed that maneuvering would need to be repeated substantially to produce the same qualitative performance as the UKF. Finally, multiple hardware-in-the-loop sequential filter tests showed strong consistency with purely simulation based tests.

Future work will focus on extending the geometric BROD algorithm by using an updated mapping between curvilinear coordinates and relative orbital elements which is valid in eccentric orbits. Additionally, a logical next step is to study the sensitivity of the sequential estimation algorithms to uncertainty in the observer orbit and attitude knowledge, and to rigorously treat that uncertainty using a consider-covariance filtering approach. With regard to modeling the solar radiation pressure perturbation, the assumption of knowing the exact differential ballistic coefficient a priori will be relaxed in future work by treating the parameter as a state variable to be estimated. Finally, the algorithms developed in this paper will be expanded to applications involving multiple cooperative observers which are fusing measurement information for enhanced distributed angles-only navigation in a multitude of relevant scenarios.

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APPENDIX A: MEASUREMENT MODEL RE-ARRANGEMENT

This appendix details the process of transforming the geometric measurement model relationship in Eq. (10) into the homogeneous system of equations in Eq. (29). The approach takes inspiration from the work of Gryzmisch and Fichter,7 who make use of a similar transformation to re-arrange the measurement equations for a linear observability assessment.

The trigonometric operators in the geometric measurement model of Eq. (10) are first inverted to yield

\[
\sin \alpha = \frac{\delta r^V_y}{\sqrt{\delta r^2_z V + \delta r^2_y V + \delta r^2_z V}} \tag{41}
\]

\[
\tan \epsilon = \frac{\delta r^V_x}{\delta r^V_z} \tag{42}
\]

Eq. (42) is re-arranged to

\[
\delta r^V_x \cos \epsilon - \delta r^V_z \sin \epsilon = 0 \tag{43}
\]

and Eq. (41) is expanded by squaring both sides and replacing \(\delta r^2_x V\) with \(\delta r^2_z V \tan^2 \epsilon\) from Eq. (42) to form

\[
\delta r^2_z V \left(1 + \tan^2 \epsilon\right) \sin^2 \alpha = \delta r^2_y V \left(1 - \sin^2 \alpha\right) \tag{44}
\]

The latter equation is further simplified by making use of the trigonometric identities \((1 + \tan^2 \epsilon) = 1/\cos^2 \epsilon\) and \((1 - \sin^2 \alpha) = \cos^2 \alpha\) to produce

\[
\delta r^V_y \cos \alpha \cos \epsilon - \delta r^V_z \sin \alpha = 0 \tag{45}
\]

Notice now that Eqs. (43) and (45) are linear homogeneous equations with respect to the VBS-frame position vector components. These two equations can be represented in matrix form as

\[
H(y(t))\delta r^V = \begin{bmatrix} \cos \epsilon & 0 \\ 0 & \cos \alpha \cos \epsilon \end{bmatrix} \begin{bmatrix} -\sin \epsilon \\ -\sin \alpha \end{bmatrix} \delta r^V = 0_{2 \times 1} \tag{46}
\]

Indeed, this is the same measurement model form leveraged for the BROD algorithm in Eq. (28).
APPENDIX B: HARDWARE-IN-THE-LOOP DETAILS AND EXAMPLE SCENE

The Optical Stimulator testbed model is shown in Figure 13a. Figure 13b shows an example far-range angles-only navigation scene generated by the testbed, with detection boxes from a simple image processing pipeline overlaid. The image processing detects regions of interest within the image using a weighted centroiding process on all pixels, and then compares groupings of categorized objects to a known star catalog that would be contained "onboard". Once the stellar objects are identified, all other objects are treated as non-stellar target objects. For this work, only the true target is simulated as a non-stellar object for simplicity.

(a) Optical Stimulator testbed, consisting of a Light Emitting Diode screen which generates scenes that are passed through a variable-focal-length collimating optic chain and measured with a vision-based sensor.

(b) Synthetic scene generated from the Optical Stimulator testbed. Stellar objects are shown in yellow squares/circles, and the non-stellar target is shown in the green square/circle.

Figure 13: The Optical Stimulator testbed, and a representative angles-only navigation synthetic scene.
REFERENCES


